THE GAUSSIAN FREE FIELD

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1. Introduction

The Gaussian free field (GFF) is a rich object from mathematical physics with many connections to many areas of probability: it is a random (generalized) function which generalizes Brownian motion to higher dimensional index sets; it is conformally invariant in the plane and in this setting its "level sets" look like Schramm-Loewner curves; it is the exponent in the metric tensor of the canonical family of random surfaces known as Liouville Quantum Gravity; the list goes on. Because of these many properties, the GFF is rich object whose basic construction and properties are even quite complicated. In this note we give a relatively self-contained construction of the GFF and and exploration of some of these fundamental properties.

We follow mainly the exposition of [5], but we also use some of the ideas and notation from [1] and [4]. For some non-trivial facts about the Dirichlet problem, spectral theory of the Laplacian, and some other facts from PDE, we use [3]. We also assume familiarity with the basic concepts of Brownian motion, functional analysis, and complex analysis.

2. Analytic Preliminaries

In this section we give the analytical preliminaries necessary for constructing the GFF. This requires defining the Green's function for domains in $\mathbb{R}^d$, and developing its relationship with the Laplacian operator and some function spaces of interest, as well as developing the spectral theory of these operators.

For this section, assume $d \geq 2$ is an integer, and that $D \subseteq \mathbb{R}^d$ a connected, open set which we call a *domain*. Usually we will assume that D is bounded; this is mostly for the sake of exposition, since many results remain true in the unbounded setting but with more complicated proofs that do not add much new insight. We remark that many important examples correspond to unbounded regions without these properties (for example, the entire space $\mathbb{R}^d$ for $d \geq 3$ and the upper half-plane $\mathbb{H}$ in $\mathbb{R}^2$ are important examples for the theory). We also emphasize, however, that the spectral theory results of the second subsection require boundedness.

Further, let us say that a point $z \in \bar{D}$ is *regular* if, for any $z \in \partial D$, when $\{B_t\}_{t\geq 0}$ denotes a standard Brownian motion in $\mathbb{R}^d$ started at z, we have $\inf\{t > 0 : B_t \notin D\} = 0$ almost surely. In other words, a is regular if a Brownian motion started at that point exits the domain in arbitrarily small times. It is clear that no element of D can be regular; we say that D is itself regular if all of its boundary points are regular. We further assume in this section that D is a regular domain, but this comes at no real cost to generality. All of the domains of interest have either differentiable boundaries or empty boundaries, and they are hence regular.

2.1. **The Green's Function.** The primary step in the construction of the analytic preliminaries is to define the Green's function $G_D : \bar{D} \times \bar{D} \to [0, \infty)$ of a regular domain D. This object encodes a lot of information about the geometry of D, and hence has many equivalent formulations, each of which is useful for different purposes. In the present work we will provide four characterizations of the Green's function, mostly without proof, so that we may use these different forms in our later construction of the GFF.

The basic definition, which amounts to our first characterization, constructs the Green's function directly in terms of the potential theory of Brownian motion. To do this, define the function $H_y : \mathbb{R}^d \to \mathbb{R}$ via $H_y(x) = -a_2^{-1} \log|x-y|$ in $d = 2$ and $H_y(x) = a_d^{-1}|x-y|^{2-d}$ for $d \geq 2$, where a_d represents the surface measure of the unit sphere $\{x \in \mathbb{R}^d : \|x\| = 1\}$. Note in particular that these are constant multiples of the unique radial harmonic functions in $\mathbb{R}^d \setminus \{0\}$.

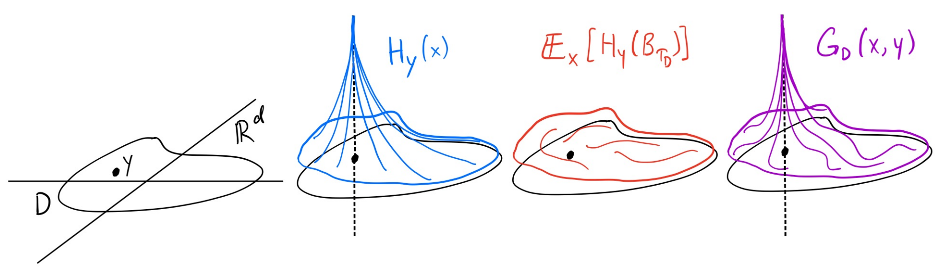
Now define a measurable space $(\Omega, \mathcal{F})$ and a stochastic process $\{B_t\}_{t\geq 0}$ which is made into a standard Brownian motion in $\mathbb{R}^d$ under the family of measures $\{\mathbb{P}_x\}_{x\in\mathbb{R}^d}$. For each $x \in \mathbb{R}^d$ write $\mathbb{E}_x$ for the expectation associated to $\mathbb{P}_x$. Also write $\tau_D = \inf\{t > 0 : B_t \notin D\}$ for the first exit time from D.

Definition 2.1. We set $G_D : \bar{D} \times \bar{D} \to \mathbb{R}$ via

$$G_D(x, y) = H_y(x) - \mathbb{E}_x[H_y(B_{\tau_D})], \tag{2.1}$$

called the *Green's function of D*.

The following cartoon illustrates the various objects leading in to the construction of the Green's function, for a regular domain $D \subseteq \mathbb{R}^d$ and any point $y \in D$:



While we focus primarily on probabilistic aspects of the Green's function in this note, we mention the Green's function is also of primary importance in PDE: The function $x \mapsto \mathbb{E}_x[H_y(B_{\tau_D})]$ is the unique solution to the Dirichlet problem $\Delta u = 0$ in D subject to the boundary condition $u|_{\partial D} = H_y$.

Next we show that the marginal functions of the Green's function are uniquely determined by a few natural properties related to these concepts. This provides our second characterization of the Green's function.

Lemma 2.2. *Fix any $y \in D$. Then, $G_D(\cdot, y)$ is the unique function $g : \bar{D} \to \mathbb{R}$ which is*

(1) *continuous on $\bar{D} \setminus \{y\}$,*
(2) *harmonic on $D \setminus \{y\}$,*
(3) *vanishing on ∂D, and*
(4) *such that $g - H_y$ is bounded.*

Proof. First let us show that $G_D(\cdot, y)$ has the enumerated properties. For (1), we note that $G_D(\cdot, y)$ is just the difference of two continuous functions. For (2), take

$x \in D$ and get $\varepsilon > 0$ such that the open ball $B_\varepsilon(x)$ satisfies $B_\varepsilon(x) \subseteq D$. Then define $\tau_{B_\varepsilon(x)} = \inf\{t > 0 : B_t \notin B_\varepsilon(x)\}$, which clearly has $\tau_{B_\varepsilon(x)} \leq \tau_D$. Recall that $B_{\tau_{B_\varepsilon(x)}}$ is uniformly distributed on $\partial B_\varepsilon(x)$ when $B_0 = x$, and write S for the surface measure on $\partial B_\varepsilon(x)$. Combining these observations with the strong Markov property and Fubini, we get:

$$\begin{aligned}\mathbb{E}_x[H_y(B_{\tau_D})] &= \mathbb{E}_x\left[\mathbb{E}\left[H_y(B_{\tau_D})|B_{\tau_{B_\varepsilon(x)}}\right]\right] \\ &= \mathbb{E}_x\left[\mathbb{E}_{B_{\tau_{B_\varepsilon(x)}}}\left[H_y(B_{\tau_D})\right]\right] \\ &= \mathbb{E}_x\left[\int_{\partial B_\varepsilon(x)} H_y(B_{\tau_D})dS(z)\right] = \int_{\partial B_\varepsilon(x)} \mathbb{E}_z[H_y(B_{\tau_D})]dS(z).\end{aligned}$$

This shows that $x \mapsto \mathbb{E}_x[H_y(B_{\tau_D})]$ has the mean-value property and is hence harmonic. Since $x \mapsto H_y(x)$ is also harmonic, this establishes (2). For (3), note that the regularity condition on D implies that we have $\tau_D = 0$ almost surely whenever $B_0 \in \partial D$, hence $G(x, y) = H_y(x) - H_y(x) = 0$ for any $x \in \partial D$. For (4), we simply note that ∂D is compact and that H_y is a continuous function on ∂D, hence it is bounded. Since $\mathbb{E}_x[H_y(B_{\tau_D})]$ is an average of these values, it satisfies the same bounds as does H_y on ∂D.

For the converse, suppose that $g_1, g_2 : \bar{D} \to \mathbb{R}$ are two functions satisfying the enumerated properties. Then for each $x \in D$, let $\{B_t\}_{t\geq 0}$ be a Brownian motion with $B_0 = x$, and define the process $\{M_t\}_{t\geq 0}$ via $M_t = g_1(B_t) - g_2(B_t)$. Since $g_1 - g_2$ is harmonic by (2), we see that $\{M_t\}_{t\geq 0}$ is a martingale. Moreover, it is continuous by (1) and the fact that a Brownian motion in $\mathbb{R}^d$ for $d \geq 2$ is not point-recurrent. By optional stopping we have $g_1(x) - g_2(x) = \mathbb{E}_x[M_0] = \mathbb{E}_x[M_{t\wedge\tau_D}]$, and since $\{M_t\}_{t\geq 0}$ is bounded by (4), we can take $\to \infty$ with bounded convergence to get $g_1(x) - g_2(x) = \mathbb{E}_x[M_{\tau_D}]$. Now by (3) we have $M_{\tau_D} = g_1(B_{\tau_D}) - g_2(B_{\tau_D}) = 0$, hence $g_1(x) = g_2(x)$. □

For our third characterization, we interpret the Green's function as the occupation density of a Brownian motion in D. To do this, write $p_t^D(x, y)$ for the subdensity at y of a Brownian motion started at x, killed when it exits D. Then we have the following, which we give without proof.

Lemma 2.3. *We have*

$$G_D(x, y) = \frac{1}{2}\int_0^\infty p_t^D(x, y)\, dt. \tag{2.2}$$

In particular, for any open set $A \subseteq D$ we have

$$\int_A G_D(x, y)\, dy = \frac{1}{2}\mathbb{E}_x\left[\int_0^{\tau_D} \mathbb{1}\{B_t \in A\}\, dt\right]. \tag{2.3}$$

Since the density p_t^D is known explicitly in some cases of interest, the characterization above can be useful for analyzing the Green's function and its asymptotics. This form also shows that G_D is a symmetric function, which is not immediately obvious from either of the prior characterizations we introduced.

For our fourth and final characterization, which is perhaps the most important, we view G_D as a kernel and study a corresponding integral operator. To do this, write $C^\infty(D)$ for the space of smooth which continuous functions $f : D \to \mathbb{R}$, and write $C_0^\infty(D)$ for the space of smooth functions $f : D \to \mathbb{R}$ with compact support in D. For $f \in C_0^\infty(D)$, let us define the function $\mathbf{G}_D f : \bar{D} \to \mathbb{R}$ via

$(\mathbf{G}_D f)(x) = \int_D G_D(x,y) f(y)\, dy$. The main step is to establish some properties of this integral operator, and then to show that it is, in a suitable sense, the inverse of the Laplacian.

To do this we must define some function spaces of interest. Write $\langle f, g\rangle_{L^2(D)} = \int_D f(x)g(x)\,dx$ for the usual inner product, and $\|f\|^2_{L^2(D)} = \int_D |f(x)|^2\,dx$ for the corresponding norm; let $L^2(D)$ denote the completion of $C_0^\infty(D)$ under this norm, which becomes a Hilbert space. Also define the bilinear form $\rangle f, g\rangle_{\mathcal{H}_0^1(D)} = \int_D \nabla f(x) \cdot \nabla g(x)\,dx$ and write $\|f\|^2_{\mathcal{H}_0^1(D)} = \int_D \|\nabla f(x)\|^2\,dx$ for the corresponding norm. The completion of $C_0^\infty(D)$ under $\|\cdot\|_{\mathcal{H}_0^1(D)}$ is a Hilbert space which we denote $\mathcal{H}_0^1(D)$. Of course, observe $\mathcal{H}_0^1(D) \subseteq L^2(D)$.

Lemma 2.4. *The integral operator of the Green's function is a continuous linear bijection $\mathbf{G}_D : L^2(D) \to \mathcal{H}_0^1(D)$, and its inverse is the map $-\Delta : \mathcal{H}_0^1(D) \to L^2(D)$.*

2.2. **Spectral Theory.** Having developed the requisite properties of the Green's function, the next step is to recall the spectral theory of the Laplacian operator and to use this theory to make some of the preceding characterizations more concrete. Recall that in the previous subsection, the assumption that the domain D was bounded was mainly to simplify some technical points; in this subsection, the assumption that D is bounded is actually necessary for the desired eigensystems to exist.

To begin, we review a standard result (see [3, Theorem 11.5.1]) on the spectral theory of the Laplacian: When D is a bounded regular domain, there exists a sequence $\{\phi_j\}_{j\in\mathbb{N}}$ of eigenfunctions of $-\Delta$ which lie in $\mathcal{H}_0^1(D) \subseteq L^2(D)$, and such that the corresponding sequence of eigenvalues $\{\lambda_j\}_{j\in\mathbb{N}}$ are positive and tend to infinity. Moreover, the sequence $\{\phi_j\}_{j\in\mathbb{N}}$ forms a complete orthonormal system in $L^2(D)$, and the sequence $\{\phi_j/\sqrt{\lambda_j}\}_{j\in\mathbb{N}}$ forms a complete orthonormal system in $\mathcal{H}_0^1(D)$. We refer to this ensemble of objects collectively as the *Laplacian eigensystem.*

Our first result gives a more concrete description of the Green's function in terms of the Laplacian eigensystem. We regard this as a certain kind of "diagonalization" result, whereby we write a complicated kernel in terms of rank-one operators.

Lemma 2.5. *For $x, y \in D$, we have*

$$G_D(x,y) = \sum_{j\in\mathbb{N}} \frac{1}{\lambda_j} \phi_j(x)\phi_j(y). \tag{2.4}$$

Proof. Fix $y \in D$, and consider the function $P(t,x) = p_t^D(x,y)$. For each $t > 0$ we can write

$$P(t,x) = \sum_{j\in\mathbb{N}} c_j(t)\phi_j(x) \tag{2.5}$$

where the sum converges in $L^2(D)$, and $\{c_j(t)\}_{j\in\mathbb{N}}$ is some sequence of real coefficients. Now recall that P also satisfies the heat equation $\partial_t P = \frac{1}{2}\Delta P$ on D subject to zero boundary conditions. In particular, we have

$$\partial_t P(t,x) = \sum_{j\in\mathbb{N}} c_j'(t)\phi_j(x)$$

$$\frac{1}{2}\Delta P(t,x) = -\sum_{j\in\mathbb{N}} \frac{c_j(t)\lambda_j}{2}\phi_j(x),$$

so matching coefficients leads to the ODE $2c_j'(t) = -c_j(t)\lambda_j$ for each $j \in \mathbb{N}$. This is solved at $c_j(t) = c_j(0)\exp(-\frac{1}{2}\lambda_j t)$, so we only need to determine the initial conditions. At this point we have shown

$$p_t^D(x,y) = \sum_{j\in\mathbb{N}} c_j(0)e^{-\frac{1}{2}\lambda_j t}\phi_j(x)\phi_j(y), \tag{2.6}$$

so applying the first identity of Lemma 2.3 and integrating over $t > 0$ gives

$$G_D(x,y) = \sum_{j\in\mathbb{N}} \frac{c_j(0)}{\lambda_j}\phi_j(x)\phi_j(y). \tag{2.7}$$

Now recall that ϕ_j is an eigenfunction of $\mathbf{G}_D$ with eigenvalue λ_j for all $j \in \mathbb{N}$, so we conclude $c_j(0) = 1$ for all $j \in \mathbb{N}$, whence the result. □

Remark 2.6. The proof of the preceding result also showed that we have

$$p_t^D(x,y) = \sum_{j\in\mathbb{N}} e^{-\frac{1}{2}\lambda_j t}\phi_j(x)\phi_j(y), \tag{2.8}$$

for all $t > 0$, which is interesting in its own right.

In fact, it can be shown that (2.4) actually holds in $L^2(D \times D)$ instead of merely pointwise, but we do not develop this idea here. We also remark that, from the form of the Green's function in the preceding result, one can derive some precise asymptotics in particular cases of interest where the eigensystem of the Laplacian is known.

Next, we note that we can characterize Sobolev space $\mathcal{H}_0^1(D)$ in terms of the Laplacian eigensystem, as

$$\mathcal{H}_0^1(D) = \left\{ \sum_{j\in\mathbb{N}} a_j \frac{\phi_j}{\sqrt{\lambda_j}} : \sum_{j\in\mathbb{N}} a_j^2 < \infty \right\} \tag{2.9}$$

where the summation is of course understood to converge in $\mathcal{H}_0^1(D)$. By making the substitution $c_j = a_j/\sqrt{\lambda_j}$, we can characterize this space equivalently as

$$\mathcal{H}_0^1(D) = \left\{ \sum_{j\in\mathbb{N}} c_j\phi_j : \sum_{j\in\mathbb{N}} \lambda_j c_j^2 < \infty \right\} \tag{2.10}$$

so $\mathcal{H}_0^1(D)$ is isometrically isomorphic to the space of real sequences $\{c_j\}_{j\in\mathbb{N}}$ satisfying $\sum_{j\in\mathbb{N}} \lambda_j c_j^2 < \infty$, under the obvious identification.

This perspective leads us to a natural generalization whereby we define, for each $r \in \mathbb{R}$, the space $\mathcal{H}_0^r(D)$ as the closure of those elements $f \in C_0^\infty(D)$ satisfying $\sum_{j\in\mathbb{N}} \lambda_j^r (\int_D f(x)\phi_j(x)\,dx)^2 < \infty$ under the norm $\|f\|^2_{\mathcal{H}_0^r(D)} = \sum_{j\in\mathbb{N}} \lambda_j^r (\int_D f(x)\phi_j(x)\,dx)^2$. The importance of this construction is that it will turn out to be useful to view the GFF as a certain random element of these generalized Sobolev spaces. In particular note that $\mathcal{H}_0^0(D) = L^2(D)$.

Finally, we remark that much is known about the asymptotics of the eigenvalues $\{\lambda_j\}_{j\in\mathbb{N}}$. The famous *Weyl's law* (see [2]) states that there exists a constant c_d depending only on the dimension d such that $\lambda_j \sim c_d \mathrm{Vol}(D) j^{2/d}$, in the sense that the ratio of the two sides tends to 1 as $j \to \infty$.

2.3. **The Case of Dimension Two.** While we have defined and studied the Green's function for regular bounded domains D in $\mathbb{R}^d$, there is a special role played by $d = 2$, in which many important symmetries are present. Again, we assume the usual conditions on D, including boundedness, although boundedness is not necessary for most of the results herein. For a first indication of where $d = 2$ appears, we give the following result which shows, in particular, that two is the only dimension in which the Green's function is scale-invariant.

Lemma 2.7. *For any $\lambda > 0$ and $x, y \in D$, we have $G_{\lambda D}(\lambda x, \lambda y) = \lambda^{2-d} G_D(x, y)$.*

Proof. For any $\lambda > 0$, set $\{B'_t\}_{t\geq 0}$ via $\lambda B'_t = B_{\lambda^2 t}$, which is just a standard Brownian motion started at x; write $\mathbb{P}'_x$ for its law and $\mathbb{E}'_x$ for the corresponding expectation. Also define $\tau'_D = \inf\{t > 0 : B'_t \notin D\}$ and note that this stopping time satisfies $\lambda^2 \tau'_D = \tau_{\lambda D}$ hence $\lambda B'_{\tau'_D} = B_{\tau_{\lambda D}}$. Thus we can compute, for $d = 2$:

$$\begin{aligned} G_{\lambda D}(\lambda x, \lambda y) &= H_{\lambda y}(\lambda x) - \mathbb{E}_{\lambda x}[H_{\lambda y}(B_{\tau_D})] \\ &= -\frac{1}{a_2} \log |\lambda x - \lambda y| - \mathbb{E}_{\lambda x}\left[-\frac{1}{a_2} \log |B_{\tau_{\lambda D}} - \lambda y|\right] \\ &= -\frac{1}{a_2} \log |\lambda x - \lambda y| - \mathbb{E}'_x\left[-\frac{1}{a_2} \log |\lambda B'_{\tau'_D} - \lambda y|\right] = G_D(x, y), \end{aligned}$$

and for $d \geq 3$:

$$\begin{aligned} G_{\lambda D}(\lambda x, \lambda y) &= H_{\lambda y}(\lambda x) - \mathbb{E}_{\lambda x}[H_{\lambda y}(B_{\tau_D})] \\ &= -\frac{1}{a_d} |\lambda x - \lambda y|^{2-d} - \mathbb{E}_{\lambda x}\left[-\frac{1}{a_d} |B_{\tau_{\lambda D}} - \lambda y|^{2-d}\right] \\ &= -\frac{1}{a_d} |\lambda x - \lambda y|^{2-d} - \mathbb{E}'_x\left[-\frac{1}{a_d} |\lambda B'_{\tau'_D} - \lambda y|^{2-d}\right] = \lambda^{2-d} G_D(x, y), \end{aligned}$$

as claimed. □

In fact, the Green's functions in dimension two have many other symmetries which we now outline. For this subsection, we consider only $d = 2$. Since the language of complex analysis is useful for simplifying certain statements in this setting, we use the natural identification of $\mathbb{R}^2$ with $\mathbb{C}$.

Recall that for two domains $D, D' \subseteq \mathbb{C}$, a *conformal map* Φ between them is a function $\Phi : D \to D'$ which is holmorphic. We say that D and D' are *conformally equivalent* if there exists a bijective conformal map from one to the other such that its inverse is also conformal; we call such a map a *conformal equivalence*. As we show next, the Green's function is not only invariant under scaling, but actually under more general conformal equivalences:

Lemma 2.8. *Suppose that $\Phi : D \to \Phi(D)$ is a conformal equivalence which extends continuously to a homeomorphism $\Phi : \bar{D} \to \overline{\Phi(D)}$. Then, $G_{\Phi(D)}(\Phi(x), \Phi(y)) = G_D(x, y)$ for all $x, y \in \bar{D}$.*

Proof. Fix $y \in D$ and define consider the function $G_{\Phi(D)}(\Phi(\cdot), \Phi(y)) : \bar{D} \to \bar{\mathbb{R}}$. We claim that $G_{\Phi(D)}(\Phi(\cdot), \Phi(y))$ satisfies the four hypotheses of Lemma 2.2; these all follow from viewing it as the composition of the functions $\Phi : \bar{D} \to \bar{D}'$ and $G_{\Phi(D)}(\cdot, \Phi(y)) : \bar{D}' \to \mathbb{R}$. Indeed, (1) follows since both functions are continuous, (2) follows since the pre-composition of a harmonic function by a conformal map is harmonic, and (3) follows since Φ maps ∂D to $\partial(\Phi(D))$ and $G_{\Phi(D)}(\cdot, \Phi(y))$ vanishes on $\partial(\Phi(D))$. For (4), note that it suffices to show that $h(x) = H_{\Phi(y)}(\Phi(x)) - H_y(x)$ is bounded on $\bar{D}$. It is clear that h is bounded on $\bar{D} \setminus \{y\}$, and it remains bounded as $x \to y$, since we can directly compute:

$$\begin{aligned} h(x) &= -\frac{1}{a_2} \log |\Phi(x) - \Phi(y)| + \frac{1}{a_2} \log |x - y| \\ &= -\frac{1}{a_2} \log \frac{|\Phi(x) - \Phi(y)|}{|x - y|} \\ &\to -\frac{1}{a_2} \log \Phi'(y). \end{aligned}$$

By Lemma 2.2, we see that $G_{\Phi(D)}(\Phi(\cdot), \Phi(y))$ and $G_D(\cdot, y)$ satisfies the enumerated properties, hence they are equal. Applying this to all $y \in D$, and noting that both functions vanish for $y \in \partial D$, gives the result. □

We remark that the preceding result, in its current form, is actually of little value since the condition about extending to a homeorphism of the closures is quite restrictive. Fortunately there exist more general results which do not suffer this curse, but we do not state them precisely here.

Next we point out that invariance under conformal equivalence (in the stronger form alluded to) is a powerful tool for calculations with the Green's function, since, if G_D is known for any particular domain D, then it can be derived for any other domain D which is conformally equivalent to it. By the Riemann mapping theorem, any two simply-connected domains are conformally equivalent, so, in principle, one can find the Green's function on any simply-connected domain just by knowing the Green's function on a particular one, say the unit disk $\mathbb{D} = \{x \in \mathbb{C} : |x| < 1\}$ We will see examples of this in the next subsection.

In fact, conformal equivalence leads one to more precise asymptotics for the rate of growth of the singularity of the Green's function near the diagonal. By property (4) of Lemma 2.2, we know $G_D(x, y) = -a_2^{-1} \log |x - y| + O(1)$ as $x \to y$, but in the planar case we have a more refined estimate:

Lemma 2.9. *Suppose that $D \subseteq \mathbb{R}^2$ is simply-connected, and fix $y \in D$. By the Riemann mapping theorem, there exists a unique conformal equivalence $\Phi : \mathbb{D} \to$ with $\Phi(0) = y$ and $\Phi'(0) > 0$. Then,*

$$G_D(x, y) = -a_2^{-1} \log |x - y| + \frac{\Phi'(0)}{2\pi} + o(1) \tag{2.11}$$

as $x \to y$. Here, $\Phi'(0)$ is called the conformal radius of D at y.

2.4. **Examples.** In this last subsection we give a few examples of the Green's function in a few domains of interest. The most important cases, as we saw in the last subsection, concern $d = 2$, so that is where we focus our attention.

First, let's find the Green's function in the upper half-plane $\mathbb{H} = \{x \in \mathbb{C} : \mathrm{Im}(x) > 0\}$. By the reflection principle, we have $p_t^{\mathbb{H}}(x, y) = p_t(x, y) - p_t(x, \bar{y})$, where p_t represents the density of a standard complex Brownian motion with no killing. Thus by Lemma 2.3 and some calculations that we omit, we have

$$G_{\mathbb{H}}(x, y) = \frac{1}{2\pi} \log \frac{|x - \bar{y}|}{|x - y|}. \tag{2.12}$$

Next we consider the unit disk $\mathbb{D} = \{x \in \mathbb{C} : |x| < 1\}$. A conformal equivalence taking $\mathbb{D}$ to $\mathbb{H}$ is known to be the Möbius transformation $\Phi(z) = \frac{1+z}{1-z}$. Thus by

(2.12) and Lemma 2.8, we get

$$G_{\mathbb{D}}(x,y) = \frac{1}{2\pi}\log\frac{|1-x\bar{y}|}{|x-y|}. \tag{2.13}$$

Like we remarked in the last section, we can boostrap our knowledge of the Green's function on $\mathbb{D}$ and $\mathbb{H}$ to knowing the function on any simply-connected domain, although we do not pursue this here.

3. The Gaussian Free Field

We are now ready to use the analytic tools of the previous section to define and study the main object of interest, the Gaussian free field (GFF). To motivate this, note that the case of $d = 1$ was exlcluded from all our previous work, although much of it could have gone through mutatis mutandis. The reason is that the GFFs in the domain $(0,1)$ and $(0,\infty)$ turn out to be nothing more than a Brownian bridge and a Brownian motion. In this way, the GFF can be seen as a generalization of these standard Brownian objects to more general domains in higher dimensions. As we will see, the GFF can be viewed as certain kind of "canonical" random function in D, although, as long $d \geq 2$, it will turn out that the GFF is not definable pointwise at all.

3.1. **The Basic Construction.** To begin, fix an integer $d \geq 2$ and let $D \subseteq \mathbb{R}^d$ be a regular domain. As in the last section, we suppose for ease of exposition that D is also bounded, but this condition is not necessary unless we want to appeal to the spectral theory results. Let $G_D : \bar{D} \times \bar{D} \to \mathbb{R}$ be the Green's function in D, as defined in the last section. Then let $\mathcal{P}(D)$ denote the space of finite signed Borel measures in D, and define $\mathcal{M}_D = \{\mu \in \mathcal{P}(D) : \int_{D\times D} |G_D(x,y)|\, d|\mu|(x)d|\mu|(y) < \infty\}$.

Our basic construction is to define the GFF as a Gaussian process indexed by $\mathcal{M}_D$. That is, it roughly describes how much (weighted) oscillation lies in the distribution $\mu \in \mathcal{M}_D$. More concretely, we need the following.

Lemma 3.1. *The function $\Sigma_D : \mathcal{M}_D \times \mathcal{M}_D \to \mathbb{R}$ defined by $\Sigma_D(\mu_1,\mu_2) = \int_{D\times D} G_D(x,y)\, d\mu_1(x)d\mu_2(y)$ is positive semidefinite.*

Proof. We need to show that for $a_1,\dots a_k \in \mathbb{R}$ and $\mu_1,\dots \mu_k \in \mathcal{M}_D$, we have $\sum_{i=1}^k \sum_{j=1}^n a_i a_j \Sigma_D(\mu_i,\mu_j) \geq 0$. By the linearity of Σ, this is equivalent to $\Sigma_D(\sum_{i=1}^n a_i\mu_i, \sum_{i=1}^n a_i\mu_i) \geq 0$, and, since $\mathcal{M}_D$ is a vector space, it suffices to show that we have $\Sigma_D(\mu,\mu) \geq 0$ for all $\mu \in \mathcal{M}_D$.

To do this, consider $f \in C_0^\infty(D)$ and note by Lemma 2.4 that the function $F = \mathbf{G}_D f \in \mathcal{H}_0^1(D)$ satisfies $\Delta F = -f$. Thus, by Fubini and Stokes' theorem we have

$$\begin{aligned}\int_{D\times D} G_D(x,y)f(x)f(y)\,dxdy &= -\int_D F(x)\Delta F(x)\,dxdy \\ &= \int_D \|\nabla F(x)\|^2\,dxdy \geq 0.\end{aligned}$$

Now we use an approximation argument to extend this to $\mu \in \mathcal{M}_D$: Choose a non-negative $C_0^\infty(D)$ function ψ with $\text{supp}(\psi) \subseteq B_1(0)$ such that $\int_D \psi(x)dx = 1$. Then for each $\varepsilon > 0$ set $\psi_\varepsilon(x) = \varepsilon^{-d}\psi(\varepsilon^{-d}x)$, and define the function $f_\varepsilon = (\mu * \psi_\varepsilon)$. It is well-known that the measures $\mu_\varepsilon(dx) = f_\varepsilon(x)dx$ satisfy $\mu_\varepsilon \to \mu$ as distributions as $\varepsilon \to 0$, and hence that

$$\int_{D\times D} G_D(x,y) f_\varepsilon(x) f_\varepsilon(y)\, dxdy \to \int_{D\times D} G_D(x,y)\, d\mu(x)d\mu(y). \tag{3.1}$$

Since the left side is non-negative for each $\varepsilon > 0$, the right must be non-negative as well, establishing the claim. ☐

Definition 3.2. The *GFF in D* is the mean-zero Gaussian process $\Gamma_D = \{\Gamma_D(\mu)\}_{\mu\in\mathcal{M}_D}$ with covariance structure $\mathrm{Cov}(\Gamma_D(\mu_1), \Gamma_D(\mu_2)) = \Sigma_D(\mu_1, \mu_2)$.

While the above construction is interesting, it is lacking in at least two ways. The first is that it appeals to an abstract result about the existence of Gaussian processes, rather than constructing the GFF from more familiar objects directly. The second is that it is perhaps more useful to view the GFF as acting a space of suitably smooth test functions rather than on a space of signed measures. In the next subsection we will allay both of these hesitations by using the eigensystem of the Laplacian.

3.2. Refining the Construction.

To enrich our understanding of the GFF past that of the basic definition, we work in a bounded domain D so that spectral decompositions can be leveraged in interesting ways. In particular, this will allow us to view the GFF in more concrete terms, both probabilistically and analytically.

For the first of our concerns, we show that there is in fact a construction of the GFF using just a sequence of independent standard Gaussian random variables.

Lemma 3.3. *If $\{Z_j\}_{j\in\mathbb{N}}$ is a sequence of independent standard Gaussian random variables, then the random variable $\Gamma_D : \mathcal{M}_D \to \mathbb{R}$ defined for $\mu \in \mathcal{M}_D$ via*

$$\Gamma_D(\mu) = \sum_{j\in\mathbb{N}} \frac{Z_j}{\sqrt{\lambda_j}} \int_D \phi_j(x)\, d\mu(x) \tag{3.2}$$

is a GFF in D. Conversely, for any GFF Γ_D in D, there exists a sequence of independent standard Gaussian random variables $\{Z_j\}_{j\in\mathbb{N}}$ such that Γ satisfies (3.2).

Proof. Note that in the setting of bounded D, the diagonalization (2.4) allows us to write the covariance kernel in the following concrete form:

$$\begin{aligned}\Sigma(\mu_1,\mu_2) &= \int_{D\times D} G_D(x,y)\, d\mu_1(x)d\mu_2(y)\\ &= \int_{D\times D}\left(\sum_{j\in\mathbb{N}} \frac{1}{\lambda_j}\phi_j(x)\phi_j(y)\right) d\mu_1(x)d\mu_2(y)\\ &= \sum_{j\in\mathbb{N}} \frac{1}{\lambda_k}\int_D \phi_j(x)\, d\mu_1(x)\int_D \phi_j(y)\, d\mu_2(y).\end{aligned}$$

This proves the first claim, since we see that defining Γ through (3.2) immediately leads to the right covariance structure. For the second claim, use the calculation above to see that, if we define the measures $\mu_j(dx) = \phi_j(x)dx$ for $j \in \mathbb{N}$, then we have $\Sigma(\mu_i, \mu_j) = \lambda_j^{-1}\mathbb{1}\{i = j\}$. In particular, the random variables

$\{\sqrt{\lambda_j}\,\Gamma_D(\mu_j)\}_{j\in\mathbb{N}}$ form a sequence of independent standard Gaussians. Then define the Gaussian process

$$\tilde{\Gamma}_D(\mu) = \sum_{j\in\mathbb{N}} \Gamma_D(\mu_j) \int_D \phi_j(x)\, d\mu(x) \tag{3.3}$$

which we will show is almost surely equal to Γ_D. Of course we have

$$\mathrm{Var}(\tilde{\Gamma}_D(\mu) - \Gamma_D(\mu)) = \mathrm{Var}(\tilde{\Gamma}_D(\mu)) + \mathrm{Var}(\Gamma_D(\mu)) - 2\mathrm{Cov}(\tilde{\Gamma}_D(\mu), \Gamma_D(\mu)), \tag{3.4}$$

and it is straightforward to compute:

$$\mathrm{Var}(\tilde{\Gamma}_D(\mu)) = \mathrm{Var}(\Gamma_D(\mu)) = \mathrm{Cov}(\tilde{\Gamma}_D(\mu), \Gamma_D(\mu)) = \Sigma(\mu, \mu)$$

from the form of the covariance kernel we derived earlier, hence $\tilde{\Gamma}_D(\mu) = \Gamma_D(\mu)$ almost surely. This finishes the proof. □

For the second concern, we ask whether Γ_D can be defined as a random process assigning real values to some family of smooth test functions rather than to signed measures. By the above result, it is tempting to try to regard Γ_D the inner product with the random element

$$\gamma_D = \sum_{j\in\mathbb{N}} \frac{Z_j}{\sqrt{\lambda_j}} \phi_j \tag{3.5}$$

in a suitable Hilbert space. It is natural to consider $\mathcal{H}_0^1(D)$, since then γ_D represents a sort of canonical Gaussian element of this space. However, the series diverges almost surely in $\mathcal{H}_0^1(D)$, since $\{\phi_j/\sqrt{\lambda_j}\}_{j\in\mathbb{N}}$ is an orthonormal system and $\sum_{j\in\mathbb{N}} Z_j^2 = \infty$ almost surely.

Not all hope is lost, however, since we can view this random sum in weaker Hilbert spaces and ask if we get convergence therein. A natural second guess is to consider this sum in $L^2(D)$, but it turns out that convergence still fails:

Lemma 3.4. *The series (3.5) diverges in $L^2(D)$ almost surely.*

Proof. Fix $A > 0$. Then define, for each $j \in \mathbb{N}$,

$$X_j = \frac{Z_j}{\sqrt{\lambda_j}} \phi_j, \tag{3.6}$$

and

$$Y_j = X_j \mathbb{1}\{\|X_j\|_{L^2(D)} \leq A\} = \frac{Z_j}{\sqrt{\lambda_j}} \phi_j \mathbb{1}\{Z_j^2 \leq A^2 \lambda_j\}. \tag{3.7}$$

When viewing these as $L^2(D)$-valued random variables, we can compute:

$$\begin{aligned}\mathrm{Var}(Y_j) &= \frac{1}{\lambda_j} \mathbb{E}[Z_j^2 \mathbb{1}\{|Z_j| \leq A\sqrt{\lambda_j}\}] \\ &= \frac{1}{\sqrt{2\pi}\lambda_j} \int_{-A\sqrt{\lambda_j}}^{A\sqrt{\lambda_j}} u^2 \exp\left(-\frac{u^2}{2}\right) du.\end{aligned}$$

Also recall the well-known asymptotic formula

$$\int_{-a}^{a} u^2 \exp\left(-\frac{u^2}{2}\right) du = \sqrt{2\pi} - O\left(\exp\left(-\frac{a^2}{2}\right)\right) \tag{3.8}$$

for $a \to \infty$, so that we have

$$\operatorname{Var}(Y_j) = \frac{1}{\lambda_j} - O\left(\exp\left(-\frac{A^2\lambda_j}{2}\right)\right). \tag{3.9}$$

Now recall by Weyl's law that we have $\lambda_j \sim c_d \operatorname{Vol}(D) j^{2/d}$ as $j \to \infty$, hence

$$\operatorname{Var}(Y_j) \sim \frac{1}{c_d \operatorname{Vol}(D) j^{2/d}} - O\left(\exp\left(-\frac{A^2 c_d^2 (\operatorname{Vol}(D))^2 j^{2/d}}{2}\right)\right). \tag{3.10}$$

The second term is summable in $j \in \mathbb{N}$, so we have $\sum_{j\in\mathbb{N}} \operatorname{Var}(Y_j) < \infty$ iff

$$\sum_{j\in\mathbb{N}} \frac{1}{j^{2/d}} < \infty, \tag{3.11}$$

and this holds true iff $d < 2$. Since we are only interested in $d \geq 2$, the sum of the variances always diverges. Finally, recall by Kolmogorov's three-series theorem that the sum of the variances converging is a necessary condition for (3.5) to converge almost surely, whence the result. □

The proof of the preceding result illustrates why we usually exclude $d = 1$ from the theory of GFF: It is the only dimension in which the GFF turns out to be a bona fide function. The "interesting" behavior occurs only when $d \geq 2$ wherein we need to make sense of the GFF as random elements of some Sobolev spaces of negative order:

Lemma 3.5. *The series (3.5) converges in $\mathcal{H}^{-s}(D)$ for all $s > \frac{d}{2} - 1$ almost surely.*

Proof. Since sums of independent random variables converge almost surely when the sum of the variances is finite, it suffices to show

$$\sum_{j\in\mathbb{N}} \operatorname{Var}\left(\frac{Z_j}{\sqrt{\lambda_j}}\phi_j\right) = \sum_{j\in\mathbb{N}} \frac{1}{\lambda_j} \|\phi_j\|^2_{\mathcal{H}_0^{-s}(D)} \tag{3.12}$$

is finite. (Here, "Var" indicates variance as $\mathcal{H}_0^{-s}(D)$-valued random vectors.) To do this, note first that we have

$$\|\phi_j\|^2_{\mathcal{H}_0^{-s}(D)} = \sum_{k\in\mathbb{N}} \lambda_k^{-s} \langle \phi_k, \phi_j \rangle_{L^2(D)} = \lambda_j^{-s}. \tag{3.13}$$

Moreover, recall by Weyl's law that we have $\lambda_j \sim c_d \operatorname{Vol}(D) j^{2/d}$ as $j \to \infty$. Hence the sum of variances is just

$$\sum_{j\in\mathbb{N}} \operatorname{Var}\left(\frac{Z_j}{\sqrt{\lambda_j}}\phi_j\right) = \sum_{j\in\mathbb{N}} \frac{1}{\lambda_j^{1+s}} \asymp \frac{1}{(c_d \operatorname{Vol}(D))^{1+s}} \sum_{j\in\mathbb{N}} \frac{1}{j^{2(1+s)/d}}. \tag{3.14}$$

Of course, this value is finite iff $2(1+s)/d > 1$, and this rearranges exactly to the stated inequality, $s > \frac{d}{2} - 1$. □

Dual to our prior comment, the proof of the preceding result illustrates why $d = 2$ is the most interesting dimension in which to study the GFF: In this setting, the GFF can be viewed as a random element of *every* Sobolev space of strictly negative order. In other words, the GFF is an element of $\mathcal{H}_0^{0-}(D)$ and hence "just shy" of being an element of $\mathcal{H}_0^0(D) = L^2(D)$.

We also explain that the GFF in any dimension has a natural type of Markov property. We have already seen that the GFF can be regarded as a generalization of Brownian motion to two or more dimensions, so it is natural to hope that some form of the temporal Markov property persists as a type of spatial Markov property, and indeed this is the case. Roughly speaking, conditional on observing Γ_D on some compact subset $A \subseteq D$, distribution of Γ_D in the domain $D \setminus A$ is a GFF $\Gamma_{D\setminus A}$ in $D \setminus A$ subject to non-zero boundary conditions at ∂A. While we do not develop this idea further, we remark that this is one of the most impotrant features of the GFF, and it can be used to show that many interesting objects (for example, large families of independent Brownian motions) are embedded inside the GFF.

3.3. **The Case of Dimension Two.** Of course, there are many other reasons that $d = 2$ is the most interesting dimension for the GFF. Primarily, these interesting properties can all be traced back to the symmetries of the Green's functions in the plane which we studied in the last section. The most important of these is a conformal invariance result.

For convenience, let us adopt the following notation. If $\Gamma = \{\Gamma(\mu)\}_\mu$ is some stochastic process indexed by a space of measures in $\mathcal{P}(D)$ and $\phi : D \to D'$ is any map bijection, write $\Gamma \circ \phi$ for the stochastic process $\{\Gamma'(\mu')\}_{\mu'}$ defined via $\Gamma'(\mu') = \Gamma(\mu' \circ \phi)$, whenever the domains of definition make sense.

Lemma 3.6. *Suppose that $\Phi : D \to \Phi(D)$ is a conformal equivalence which extends continuously to a homeomorphism $\Phi : \bar{D} \to \overline{\Phi(D)}$. If Γ_D is a GFF in D, then the $\Gamma_D \circ \Phi$ is a GFF in $\Phi(D)$.*

Proof. Write $\Gamma' = \{\Gamma'(\mu')\}_{\mu' \in \mathcal{M}_{\Phi(D)}} = \Gamma_D \circ \Phi$, so that we have $\Gamma'(\mu') = \Gamma_D(\mu' \circ \Phi)$ for all $\mu' \in \mathcal{M}_{\Phi(D)}$. By Lemma 2.8 and a simple change of variables, we get

$$\begin{aligned}
\mathrm{Cov}(\Gamma'(\mu_1'), \Gamma'(\mu_2')) &= \mathrm{Cov}(\Gamma_D(\mu_1' \circ \Phi), \Gamma_D(\mu_2' \circ \Phi)) \\
&= \int_{D\times D} G_D(x, y)\, d(\mu_1' \circ \Phi)(x) d(\mu_2' \circ \Phi)(x) \\
&= \int_{D\times D} G_{\Phi(D)}(\Phi(x), \Phi(y))\, d(\mu_1' \circ \Phi)(x) d(\mu_2' \circ \Phi)(x) \\
&= \int_{\Phi(D)\times\Phi(D)} G_{\Phi(D)}(\Phi(x), \Phi(y))\, d\mu_1'(x') d\mu_2'(y')
\end{aligned}$$

so Γ' indeed has the correct covariance structure. □

As we remarked before, this is result is of little value in its current form, but stronger true statements can be easily derived from analogous stronger true statements about conformal invariance for the Green's function.

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