LOCAL TIMES OF CONTINUOUS SEMIMARTINGALES

Contents

1. Introduction

1.1. **Motivation.** How much time does a stochastic process spend in a given state? If the state space is discrete, then this quantity is easy to define and compute. However, when the state space is continuous (say, all of $\mathbb{R}$), then it is not clear how to even define such an object. The theory of local times is a way to make this idea precise for continuous semimartingales.

Heuristically, the local time of a continuous one-dimensional semimartingale $X = \{X_t\}_{t\geq 0}$ around a point a is the "amount of time" X spends at a. One candidate for this value is the random measure

$$\int_0^t \delta(X_s - a) d\langle X, X\rangle_s, \tag{1}$$

if a Lebesgue-Stieljes integral over measure measures can be made meaningful. Perhaps another candidate is the limit

$$\lim_{\varepsilon\to 0} \frac{1}{2\varepsilon} \int_0^t \mathbf{1}_{\{X_s \in (a-\varepsilon, a+\varepsilon)\}} d\langle X, X\rangle_s, \tag{2}$$

if it can be shown to exist in some suitable sense. As we develop the theory of local times in these notes, we will see that both of these intuitions can be made precise.

Note in particular that in the above we are always integrating with $d\langle X, X\rangle_s$ instead of with ds. An explanation for this quirk is that the increasing process $\{\langle X, X\rangle_t\}_{t\geq 0}$ is in some sense the "natural time-scale" for the process X. Of course, for Brownian motion this reduces to the usual time-scale.

We will first explore the general theory of local times of continuous semimartingales, then we will focus on the particular case of local times of Brownian motion in which the abstract theory can be made very concrete. Then we'll see a classical application of local times to SDEs.

1.2. **Preliminaries Convex Functions.** In this subsection we'll review some important results about convex functions on the real line. These are important since, as we will later see, the theory of local times can be in part viewed as the study of relaxing Ito's lemma from C^2 functions to convex functions.

Let $f : \mathbb{R} \to \mathbb{R}$ be any convex function. If f is convex and twice-differentiable, then we know from basic calculus that its derivative f' is non-decreasing and that its second derivative f'' is non-negative. Our first goal is to establish a generalization of these properties when f is convex but no further assumptions are placed on it.

The definition of convexity is that, for any $a, b \in \mathbb{R}$ and $t \in [0, 1]$ we have $f(ta+(1-t)b) \leq tf(a)+(1-t)f(b)$. Now take any $y_1 \leq y_2 < x$ in $\mathbb{R}$ and apply the definition of convexity with $a = y_1, b = x$, and $t = (x - y_2)/(x - y_1)$, which gives

$$f(y_2) \leq \frac{x - y_2}{x - y_1} f(y_1) + \frac{(x - y_1) - (x - y_2)}{x - y_1} f(x), \tag{3}$$

and rearranging this yields

$$\frac{f(x) - f(y_1)}{x - y_1} \leq \frac{f(x) - f(y_2)}{x - y_2}. \tag{4}$$

This shows that, for fixed x, the difference quotient of f near x is non-decreasing as $y \uparrow x$. In particular, the left-hand derivative of f is well-defined at every point in $\mathbb{R}$, and we denote this function by f'_-. A similar calculation shows that a similar bound is true for points to the right of x, and hence that the right-hand derivative f'_+ is also well-defined at every point. Combining both bounds shows that f is locally Lipschitz and hence locally bounded.

Next we'll show that the function f'_- is left-continuous. Suppose $y \uparrow x$ and $h \downarrow 0$. Now use the fact that convex functions are continuous, the fact that non-decreasing limits can be interchanged, and the definition of f'_- to check:

$$\begin{aligned}
\lim_{h \downarrow 0} f'_-(x - h) &= \lim_{h \downarrow 0} \lim_{y \uparrow x} \frac{f(x - h) - f(y - h)}{x - y} \\
&= \lim_{y \uparrow x} \lim_{h \downarrow 0} \frac{f(x - h) - f(y - h)}{x - y} \\
&= \lim_{y \uparrow x} \frac{f(x) - f(y)}{x - y} \\
&= f'_-(x)
\end{aligned}$$

as claimed. Another easy calculation shows that f'_- is non-decreasing.

Next we consider the "second derivative" of f. Of course, this does not exist in the usual sense, but it may be possible to define a weak derivative in the sense of distributions. To do this we need to check that f is locally integrable, and this follows from the fact that it is locally bounded as we showed in (3). Now recall that the weak second derivative of f is the distribution sending $\phi \mapsto \int_{\mathbb{R}} \phi''(x) f(x) dx$ for $\phi \in C_c^\infty(\mathbb{R})$; we denote this by f'' and we write $\langle f'', \phi \rangle = \int_{\mathbb{R}} \phi''(x) f(x) dx$.

Let us understand what object the second derivative is. For any test function $\phi \in C_c^\infty(\mathbb{R})$, we have by definition:

$$\langle f'', \phi \rangle = \int_{\mathbb{R}} \phi''(x) f(x) dx = \int_{\mathbb{R}} \lim_{h \downarrow 0} \left(\frac{\phi'(x+h) - \phi'(x)}{h} \right) f(x) dx. \tag{5}$$

Now note that the limit converges for any $x \in \mathbb{R}$. Moreover, the difference quotient is zero outside of some compact set containing $\text{supp}(\phi)$. So we can interchange the limit and integral by dominated convergence. Then change variables and use dominated convergence again to get:

$$\begin{aligned}
\langle f'', \phi \rangle &= \int_{\mathbb{R}} \lim_{h \downarrow 0} \left(\frac{\phi'(x+h) - \phi'(x)}{h} \right) f(x) dx \\
&= \lim_{h \downarrow 0} \frac{1}{h} \left(\int_{\mathbb{R}} \phi'(x+h) f(x) dx - \int_{\mathbb{R}} \phi'(x) f(x) dx \right) \\
&= \lim_{h \downarrow 0} \frac{1}{h} \left(\int_{\mathbb{R}} \phi'(x) f(x-h) dx - \int_{\mathbb{R}} \phi'(x) f(x) dx \right) \\
&= \int_{\mathbb{R}} \phi'(x) \lim_{h \downarrow 0} \left(\frac{f(x-h) - f(x)}{h} \right) dx \\
&= - \int_{\mathbb{R}} \phi'(x) f'_{-}(x) dx
\end{aligned}$$

Note that the integral on the right is just the Lebesgue-Stieljes integral of f'_{-} against the integrator ϕ'', which has bounded variation since ϕ''' is continuous and compactly supported. So, applying integration by parts yields

$$\langle f'', \phi \rangle = \int_{\mathbb{R}} \phi''(x) df'_{-}(x) \tag{6}$$

Since f'_{-} is a non-decreasing function, the Lebesgue-Stieljes integral coincides with the Lebesgue integral with respect to a (non-negative) Radon measure. Intuitively, this could be written as $df'_{-}(x)$ but for notation's sake we will instead write this as $f''(dx)$. The fact that the distributional second derivative of a convex function f is a non-negative Radon-measure can be seen as a generalization of the fact that second derivatives of convex functions are non-negative.

Moving on, we present a useful decomposition of convex functions that will be important later. Suppose that $I \subseteq \mathbb{R}$ is some compact interval and that $f : I \to \infty$ is convex. We claim that there exists a unique pair of reals α, β depending on f and I such that

$$f(x) = \alpha x + \beta + \frac{1}{2} \int_I |x - a| f''(da) \tag{7}$$

holds for all $x \in I$. If I consists of a single point then this is obvious. If I has at least two points x_1, x_2, then the coefficients α, β must satisfy the linear system

$$\begin{pmatrix} f(x_1) - \frac{1}{2} \int_I |x_1 - a| f''(da) \\ f(x_2) - \frac{1}{2} \int_I |x_2 - a| f''(da) \end{pmatrix} = \begin{pmatrix} x_1 & 1 \\ x_2 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \tag{8}$$

which admits a unique solution. Next note that this decomposition is related to the left-hand derivative via the identity:

$$f'_{-}(x) = \frac{1}{2}\int_I \operatorname{sign}(x-a) f''(da) + \alpha \tag{9}$$

for all x in the interior of I, where sign is defined to be 1 on $(0,\infty)$ and -1 on $(-\infty, 0]$. To see this, note that the measure $f''(da)$ is finite on the compact interval I, and that the difference quotients are uniformly bounded on I. So, dominated convergence and the fact that the left-hand derivative of $x \mapsto |x-a|$ is $\operatorname{sign}(x-a)$ gives:

$$\begin{aligned} f'_{-}(x) &= \lim_{y\uparrow x} \frac{f(x)-f(y)}{x-y} \\ &= \lim_{y\uparrow x} \frac{1}{x-y}\int_I (|x-a|-|y-a|) f''(da) \\ &= \int_I \lim_{y\uparrow x} \frac{|x-a|-|y-a|}{x-y} f''(da) = \frac{1}{2}\int_I \operatorname{sign}(x-a) f''(da), \end{aligned}$$

which will be a useful identity later.

In the above we have collected useful facts about convex functions in complete generality, and we have seen that many properties of non-smooth convex functions can be understood in analogy with smooth convex functions. However, sometimes this framework is still not enough, and we will instead need to applying a "smoothing" operation to some non-smooth convex functions.

Let $j : \mathbb{R} \to \mathbb{R}$ denote a compactly-supported C^∞ function with $\operatorname{supp}(j) \subseteq (-\infty, 0]$ and $\int_{-\infty}^0 j(y)dy = 1$, and define j_n via $j_n(y) = nj(ny)$. We can think of $\{j_n\}$ as a sequence of smooth functions which "approximate the dirac delta" in the sense of distributions. Now if f is any convex function, define $f_n(x) = \int_{-\infty}^0 f(x+y) j_n(y)dy$. Some important facts that we will (but will not prove, for brevity's sake) is that each f_n is a C^∞ function, that f_n converges to f pointwise, and that f'_n converges upwards to f'_{-}.

2. General Local Times

In this section we construct the local time of a continuous semimartingale in a rigorous way and explore many of its general properties. Throughout this section let $X = \{X_t\}_{t\geq 0}$ be a continuous semimartingale on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ which is adapted to a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual completeness conditions. We will usually leave the probability space and filtration "in the background" and we won't reference them unless it's necessary.

2.1. As a Temporal Process.

Our first step is to rigorously construct the local time of a continuous semimartingale. This will entail first taking the perspective that some point $a \in \mathbb{R}$ is fixed and that we wish to study how the process $\{X_t\}_{t\geq 0}$ accumulates near a as time varies.

Theorem 2.1 (Tanaka's Formula)**.** For each $a \in \mathbb{R}$, there exists a continuous, non-decreasing, adapted process L^a_t, called the *local time of X at a*, such that

$$|X_t - a| = |X_0 - a| + \int_0^t \operatorname{sign}(X_s - a) dX_s + L^a_t \tag{10}$$

holds almost surely. In particular, $|X_t - a|$ is a semimartingale.

Proof. We first prove a slightly more general statement. If $f : \mathbb{R} \to \mathbb{R}$ is any convex function, we claim that there exists a continuous, non-decreasing process $\{A_t^f\}_{t\geq 0}$ such that

$$f(X_t) = f(X_0) + \int_0^t f'_-(X_s)dX_s + \frac{1}{2}A_t^f \tag{11}$$

holds almost surely for all $t \geq 0$.

To do this, set $T_k = \inf\{t \geq 0 : X_t \notin [-k, k]\}$, and note that the stopped process $\{X_t^{T_k}\}_{t\geq 0}$ is bounded. Now for any convex $f : \mathbb{R} \to \mathbb{R}$, we can get a sequence $\{f_n\}$ of C^∞ functions with $f_n \to f$ pointwise $f'_n \uparrow f'_-$. Applying Ito's formula to the function f_n, we get

$$f_n(X_t^{T_k}) = f_n(X_0^{T_k}) + \int_0^t f'_n(X_s^{T_k})dX_s^{T_k} + \frac{1}{2}\int_0^t f''_n(X_s^{T_k})d\langle X^{T_k}, X^{T_k}\rangle_s \tag{12}$$

so the claim holds for each n via $A_t^{f_n} = \int_0^t f''_n(X_s^{T_k})d\langle X^{T_k}, X^{T_k}\rangle_s$.

As $n \to \infty$, we have $f_n(X_t^{T_k}) \to f(X_t^{T_k})$ almost surely for any fixed $t \in [0, T_k]$. Taking the intersection over rationals gives that $f_n(X_t) \to f(X_t)$ holds for all $t \in [0, T_k] \cap \mathbb{Q}$ almost surely. Now note that, we have $f'_n(X_s^{T_k}) \to f'_-(X_s^{T_k})$ for all $s \in [0, T_k]$ almost surely and also that $f'_n(X_s^{T_k})$ is uniformly bounded above, since f_n is continuous and X^{T_k} is bounded. Hence, the dominated convergence theorem for stochastic integrals shows that the integral on the right side converges to $\int_0^t f'_-(X_s^{T_k})dX_s^{T_k}$ uniformly on $t \in [0, T_k]$ in probability. In particular, there exists a subsequence $\{n_j\}$ which almost surely converges uniformly on $[0, T_k]$. Hence, with probability one, the sequence $\{A_t^{f_{n_1}}\}_{t\in[0,T_k]}, \{A_t^{f_{n_2}}\}_{t\in[0,T_k]}, \ldots$ converges pointwise to a limit which we denote $\{A_t^f\}_{t\in[0,T_k]}$. Since each $\{A_t^{f_{n_j}}\}_{t\in[0,T_k]}$ is non-decreasing, this implies that $\{A_t^f\}_{t\in[0,T_k]}$ is also non-decreasing. Now use that $T_k \to \infty$ holds almost surely to get that $\{A_t^f\}_{t\geq 0}$ is as desired.

Now choose $a \in \mathbb{R}$ and let us apply the result to the convex functions $x \mapsto (x-a)^+$ and $x \mapsto (x-a)^-$. We get non-decreasing processes $\{A_t^+\}_{t\geq 0}$ and $\{A_t^-\}_{t\geq 0}$ satisfying

$$(X_t - a)^+ = (X_0 - a)^+ + \int_0^t \mathbf{1}_{(a,\infty)}(X_s)dX_s + \frac{1}{2}A_t^+$$
$$(X_t - a)^- = (X_0 - a)^- - \int_0^t \mathbf{1}_{(-\infty,a]}(X_s)dX_s + \frac{1}{2}A_t^-$$

almost surely. Adding these together, we get:

$$|X_t - a| = |X_0 - a| + \int_0^t \operatorname{sign}(X_s - a)dX_s + \frac{1}{2}\left(A_t^+ + A_t^-\right) \tag{13}$$

almost surely, so the claim holds for $L_t^a = \frac{1}{2}(A_t^+ + A_t^-)$.

□

We have established the existence of a process $\{L_t^a\}_{t\geq 0}$ for each $a \in \mathbb{R}$ but at this point it may be a bit mysterious. To see that this process deserves to be called a local time, we make the following heuristic calculation: The function $f(x) = |x-a|$ has left-hand derivative $f'_-(x) = \text{sign}(x-a)$ and distributional second derivative $f''(x) = \delta_{x-a}$. So, an application of Ito's formula gives $L_t^a = \int_0^t \delta_{X_s - a} d\langle X, X\rangle_s$, exactly as we prediced in (1).

For each $a \in \mathbb{R}$, the process $\{L_t^a\}_{t\geq 0}$ is non-decreasing, hence of bounded variation. So we can consider the Lebesgue-Stieljes measure with respect to this process, and this defines a measure on the non-negative real line, which we denote dL_t^a. In some sense, the measure dL_t^a encodes the amount of time spent at a by the semimartingale X. So, the following result is intuitive:

Lemma 2.2. For each $a \in \mathbb{R}$, we have $\text{supp}(dL_t^a) \subseteq \{t \geq 0 : X_t = a\}$ almost surely.

Proof. Since $(X_t - a)^2 = |X_t - a|^2$, we have two ways to compute this common value: use Ito's formula for $f(x) = x^2$ on the semimartingale $X_t - a$, or use Ito's formula for $f(x) = x^2$ on $|X_t - a|$ and then apply Tanaka's formula. From the first method, we have:

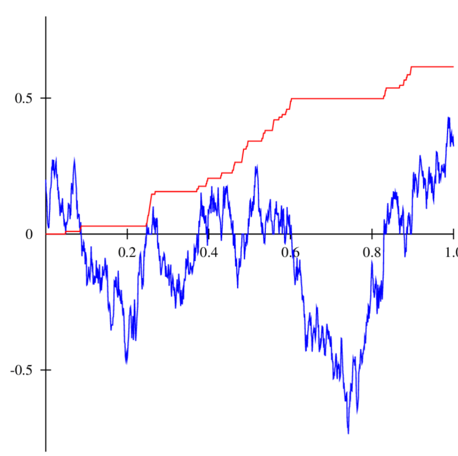
$$(X_t - a)^2 = (X_0 - a)^2 + 2\int_0^t (X_s - a) dX_s + \langle X, X\rangle_s. \tag{14}$$

And from the second method, we have:

$$\begin{aligned}|X_t - a|^2 &= |X_0 - a|^2 + 2\int_0^t |X_s - a| d(|X_{\cdot} - a|)_s + \langle |X_{\cdot} - a|, |X_{\cdot} - a|\rangle_s \\ &= |X_0 - a|^2 + 2\int_0^t |X_s - a|\text{sign}(X_s - a) dX_s + 2\int_0^t |X_s - a| dL_s^a + \langle |X_{\cdot} - a|, |X_{\cdot} - a|\rangle_s.\end{aligned}$$

Now observe that we have both $X_s - a = |X_s - a|\text{sign}(X_s - a)$ and $\langle |X_{\cdot} - a|, |X_{\cdot} - a|\rangle_s = \langle X, X\rangle_s$. So, the two calculations above give $\int_0^t |X_s - a| dL_s^a = 0$, which implies $\text{supp}(dL_t^a) \subseteq \{t \geq 0 : X_t = a\}$. Since everything above holds almost surely, we get the conclusion almost surely. $\square$

To get some intuition for the idea we just developed, consider the following image that traces a sample path of a Brownian motion and its corresponding local time at zero:



2.2. **As a Spatial Process.** Now we change focus slightly, instead focusing on the perspective that some time $t \geq 0$ is fixed and we wish to study the distribution of points that the process X has accumulated near.

It is not surprising that the answer to this new question will come in the form of turning L into a sort of "occupation measure" for X. Of course, there must be certain regularity properties satisfied for this to make sense. Without going into the details, we give the following important result:

Lemma 2.3. There exists a process $\{\tilde{L}_t^a\}$ which, when viewed as map $\mathbb{R} \times \Omega \to [0, \infty)$ for fixed $t \geq 0$, is $\mathcal{B}(\mathbb{R}) \otimes \mathcal{F}$-measurable and such that, for every $a \in \mathbb{R}$, the process $\{\tilde{L}_t^a\}_{t \geq 0}$ is a modification of $\{L_t^a\}_{t \geq 0}$.

Throughout the remainder of this section, assume that L represents this measurable modification of the local time which we previously constructed.

Theorem 2.4 (Occupation Formula)**.** If $\{X_t\}$ is a continuous semimartingale with local time $\{L_t^a\}_{t \geq 0, a \in \mathbb{R}}$, then, almost surely, we have

$$\int_0^t g(X_s) d\langle X, X\rangle_s = \int_{\mathbb{R}} g(a) L_t^a da \tag{15}$$

for every t and for every positive measurable function $g : \mathbb{R} \to [0, \infty)$.

Proof. First let $f : \mathbb{R} \to \mathbb{R}$ be any convex function, and consider the function $f_n = f|_{K_n}$ for $K_n = [-n, n]$. On the set K_n, get constants $\alpha_n, \beta_n \in \mathbb{R}$ such that the identity

$$f(x) = \alpha_n x + \beta_n + \frac{1}{2} \int_{K_n} |x - a| f''(da) \tag{16}$$

holds for all $x \in K_n$. Now consider the stopped process $\{X_t^{T_n}\}_{t \geq 0}$ up to time $T_n = \inf\{t \geq 0 : X_t \notin K_n\}$. We can use Tanaka's formula to write

$$\begin{aligned} f(X_t^{T_n}) &= \alpha_n X_t^{T_n} + \beta_n + \frac{1}{2} \int_{K_n} |X_t^{T_n} - a| f''(da) \\ &= \alpha_n X_t^{T_n} + \beta_n + \frac{1}{2} \int_{K_n} |X_0^{T_n} - a| f''(da) + \frac{1}{2} \int_{K_n} \int_0^t \operatorname{sign}(X_s^{T_n} - a) dX_s^{T_n} f''(da) + \frac{1}{2} \int_{K_n} L_t^a f''(da) \\ &= \alpha_n (X_t^{T_n} - X_0^{T_n}) + f(X_0) + \frac{1}{2} \int_{K_n} \int_0^t \operatorname{sign}(X_s^{T_n} - a) dX_s^{T_n} f''(da) + \frac{1}{2} \int_{K_n} L_t^a f''(da). \end{aligned}$$

Now let us inspect the double integral term. Since the sign function is bounded and $f''(da)$ is a finite measure on K_n, we can apply Fubini's theorem for stochastic integrals to change the order of integration. Then use the identity

$$\frac{1}{2} \int_{K_n} \operatorname{sign}(X_s^{T_n} - a) f''(da) = f'_-(X_s^{T_n}) - \alpha_n \tag{17}$$

to get

$$\begin{aligned}\frac{1}{2}\int_{K_n}\int_0^t \operatorname{sign}(X_s^{T_n}-a)dX_s^{T_n}f''(da) &= \frac{1}{2}\int_0^t (f'_-(X_s^{T_n})-\alpha_n)dX_s^{T_n} \\ &= \frac{1}{2}\int_0^t f'_-(X_s^{T_n})-\alpha(X_t^{T_n}-X_0^{T_n}).\end{aligned}$$

Plugging this into the equation above gives

$$f(X_t^{T_n}) = f(X_0^{T_n}) + \int_0^t f'_-(X_s^{T_n})dX_s^{T_n} + \frac{1}{2}\int_{K_n} L_t^a f''(da). \tag{18}$$

Now consider taking $n\to\infty$. Note that the second integral on the right is just a Lebesgue integral (with a random integrand) and that $K_n \uparrow \mathbb{R}$, so monotone convergence applies. Moreover, for the remaining terms we note that $T_n\to\infty$ holds almost surely. Thus, we get that

$$f(X_t) = f(X_0) + \int_0^t f'_-(X_s)dX_s + \frac{1}{2}\int_{\mathbb{R}} L_t^a f''(da). \tag{19}$$

holds almost surely. This can be viewed as a sort of generalization of Ito's formula to convex functions, and is sometimes referred to as the Ito-Tanaka formula.

Now we prove the main claim. For $p,q,\varepsilon\in\mathbb{Q}$ with $p<q$ and $\varepsilon\ >0$, let $g_{p,q,\varepsilon}$ denote the function which is equal to zero on $(-\infty,p-\varepsilon]\cup[q+\varepsilon,\infty)$, equal to one on $[p,q]$, and linear in between these values. Then $g_{p,q,\varepsilon}$ is the second derivative of some convex C^2 function, namely $f_{p,q,\varepsilon}(x)=\int_{\mathbb{R}}|x-a|g_{p,q,\varepsilon}(a)da$, and $f''_{p,q,\varepsilon}(da)$ is just $g_{p,q,\varepsilon}(a)da$. Now we can apply both Ito's formula and the Ito-Tanaka formula to get:

$$\begin{aligned}f_{p,q,\varepsilon}(X_t) &= f_{p,q,\varepsilon}(X_0) + \int_0^t (f_{p,q,\varepsilon})'_-(X_s)dX_s + \frac{1}{2}\int_0^t g_{p,q,\varepsilon}(X_s)d\langle X,X\rangle_s \\ &= f_{p,q,\varepsilon}(X_0) + \int_0^t (f_{p,q,\varepsilon})'_-(X_s)dX_s + \frac{1}{2}\int_{\mathbb{R}} L_t^a g_{p,q,\varepsilon}(a)da.\end{aligned}$$

Since the terms on the right side of each line must agree, this shows that the occupation formula holds for $g_{p,q,\varepsilon}$ almost surely. Since there are only countably many triples (p,q,ε) as above, we conclude that, with probability one, the occupation formula holds for all functions in the collection $\mathcal{G}=\{g_{p,q,\varepsilon}: p,q,\varepsilon\in\mathbb{Q}, p<q\}$.

On this event of probability one, let $\mathcal{H}$ denote the collection of all measurable functions $g:\mathbb{R}\to\mathbb{R}$ for which the occupation formula holds. By monotone convergence, we see that $\mathcal{H}$ is closed under pointwise non-decreasing limits. Consequently, since elements of $\mathcal{G}$ can approximate open sets from below, the previous observation gives $\mathbf{1}_U\in\mathcal{H}$ for all open sets $U\subseteq\mathbb{R}$. Also, it is easy to see that satisfaction of the occupation times formula is closed under finite sums and by multiplication by scalars. Therefore, the monotone class theorem shows that $\mathcal{H}$ contains all bounded measurable functions. But applying monotone convergence for the positive and negative parts separately, this implies that $\mathcal{H}$ contains all measurable functions, hence the claim is proved. □

A consequence of the occupation formula is that Ito's formula holds for some functions that are not C^2, namely for functions $f : \mathbb{R} \to \mathbb{R}$ that are twice differentiable with f'' locally bounded. Strictly speaking this could have been proved with the same monotone class argument that we uesd at the end of the proof of the occupation formula.

2.3. **As a Space-Time Process.** As our last step, we consider L as a function of both variables t and a simultaneously. In a sense, this means we can consider the random field of local times $\{L_t^a\}_{t\geq 0, a\in\mathbb{R}}$. We state but do not prove the next result:

Theorem 2.5. For any continuous semimartingale X, there exists a modification of $\{L_t^a\}_{t\geq 0, a\in\mathbb{R}}$ which, almost surely, is continuous in t and cadlag in a. If X is a local martingale, then the modification can be taken to be jointly continuous in (t, a).

Throughout the remainder of the talk, we assume that L represents the modification guaranteed above.

We also remark that much finer statements about the continuity of L are understood than just what is written above. For example, it can be shown that, for a local martingale X, the smoothness of L in a is Hölder continuous with exponent α for all $\alpha < \frac{1}{2}$. Moreover, the quadratic variation of the process $a \mapsto L_t^a$ can, in some cases, by written rather explicitly.

Our last result of this section is to justify that our definition of local time also coincides with the second intuitive notion we came up with at the beginning of the talk. Specifically, we want to show that the "limiting occupation time" gives the local time.

Lemma 2.6. If X is a continuous semimartingale, then, almost surely, we have

$$L_t^a = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_0^t \mathbf{1}_{[a,a+\varepsilon)}(X_s) d\langle X, X\rangle_s \tag{20}$$

for all $a \in \mathbb{R}$ and $t \geq 0$. If X is a continuous local martingale, then, almost surely, we have

$$L_t^a = \lim_{\varepsilon\to 0} \frac{1}{2\varepsilon} \int_0^t \mathbf{1}_{(a-\varepsilon,a+\varepsilon)}(X_s) d\langle X, X\rangle_s \tag{21}$$

for all $a \in \mathbb{R}$ and $t \geq 0$.

Proof. For the first claim, apply the occupation times formula with $g(x) = \mathbf{1}_{[a,a+\varepsilon)}$ to get

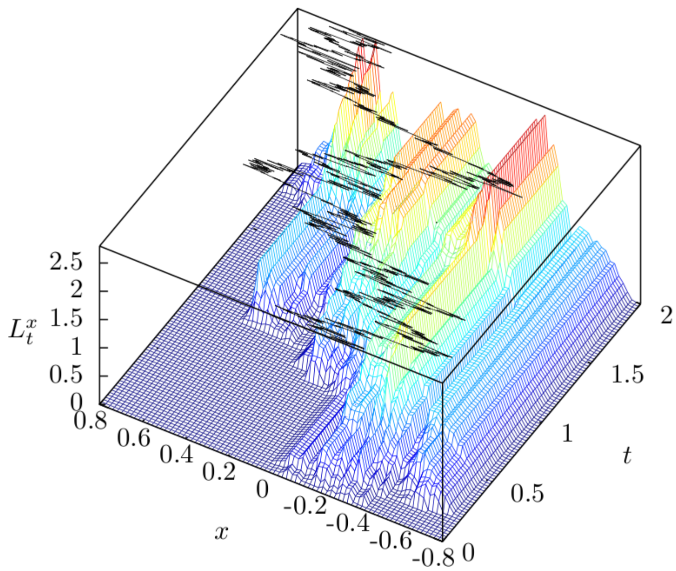
$$\int_0^t \mathbf{1}_{[a,a+\varepsilon]}(X_s) d\langle X, X\rangle_s = \int_{\mathbb{R}} \mathbf{1}_{[a,a+\varepsilon]}(a') L_t^{a'} da'. \tag{22}$$

By the preceding theorem, L_t^a is almost surely right-continuous at a, so the fundamental theorem of calculus for Lebesgue integrals gives

$$\frac{1}{\varepsilon} \int_0^t \mathbf{1}_{[a,a+\varepsilon]}(X_s) d\langle X, X\rangle_s = \frac{1}{\varepsilon} \int_{\mathbb{R}} \mathbf{1}_{[a,a+\varepsilon]}(a') L_t^{a'} da' \to L_t^a, \tag{23}$$

as claimed. The second claim follows by the same argument, using the second part of the preceding result. □

To visualize this space-time process, the image below is a simulation of a sample path of a Brownian motion and of the correspond field of local times:



3. Brownian Local Times

In this section we specialize the theory above to the case of the one-dimensional Brownian motion. As we will see, many of the abtract results above can be made much more concrete in this setting.

Moreover, we must make a historical point: Most of the theory of local times was first developed for Brownian motion, and the generalization to continuous semimartingales came later. This explains why some straightforward definitions and corollaries appear as named theorems and formulas in our presentation here.

Throughout this section we will focus mainly on the local time at zero, so we write L_t in place of L_t^0 unless when we need to emphasize zero specifically. Also, the underlying probability space and filtration will play more of a role in this conversation; let $(\Omega, \mathcal{F}, \mathbb{P})$ denote the probability space on which a Brownian motion $B = \{B_t\}_{t\geq 0}$ is defined, and let $\{\mathcal{F}_t\}_{t\geq 0}$ denote a filtration of $\mathcal{F}$ to which B is adapted. For any process $X = \{X_t\}_{t\geq 0}$, let $\{\mathcal{F}_t^X\}_{t\geq 0}$ denote the natural filtration of X, defined via $\mathcal{F}_t^X = \sigma(X_s : s \leq t)$.

As a first observation, note that Tanaka's formula immediately implies that we have the following identity:

$$|B_t| = \int_0^t \operatorname{sign}(B_s) dB_s + L_t. \tag{24}$$

The $\{\mathcal{F}_t^B\}_{t\geq 0}$-adapted local martingale $\beta_t = \int_0^t \operatorname{sign}(B_s) dB_s$ satisfies $\langle \beta_t, \beta_t \rangle = \int_0^t (\operatorname{sign}(B_s))^2 ds = t$, so by Levy's characterization it is a $\{\mathcal{F}_t^B\}_{t\geq 0}$-Brownian motion. However, by the limiting occupation time lemma, we also have

$$L_t = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_0^t \mathbf{1}_{(-\varepsilon,\varepsilon)}(B_s) ds = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_0^t \mathbf{1}_{[0,\varepsilon)}(|B_s|) ds. \tag{25}$$

This proves that $\{L_t\}_{t\geq 0}$ is adapted to $\{\mathcal{F}_t^{|B|}\}_{t\geq 0}$, hence that $\{\beta_t\}_{t\geq 0}$ is adapted to $\{\mathcal{F}_t^{|B|}\}_{t\geq 0}$. In particular, this gives $\mathcal{F}_t^{\beta} \subseteq \mathcal{F}_t^{|B|}$ for all $t \geq 0$.

We can get an even more refined understanding of the relationship between $|B|, \beta$, and L with a little more work. To establish this, we will need the following deterministic lemma:

Lemma 3.1 (Skorokhod)**.** Let $y : [0,\infty) \to \mathbb{R}$ be a continuous function with $y(0) \geq 0$. Then the functions $a, z : [0,\infty) \to \mathbb{R}$ defined as

$$a(t) = \sup_{s\in[0,t]} y^-(s) \qquad \text{and} \qquad z(t) = y(t) + a(t) \tag{26}$$

are the unique pair satisfying

(1) $z = y + a$,
(2) z is non-negative,
(3) a is non-decreasing, continuous, vanishing at zero, and the Lebesgue-Stieljes measure da has support contained in the set $\{t \geq 0 : z(t) = 0\}$.

Proof. The functions a and z satisfy (1) by construction. Also (2) holds because any $t \geq 0$ with $y(t) \leq 0$ implies $a(t) \geq -y(t)$. For (3), it is clear that a is non-decreasing, continuous, and vanishing at zero. To see the claim about the support of the measure $da(t)$, suppose that $t \geq 0$ is any point with $a(t+\varepsilon) - a(t-\varepsilon) > 0$ for all $\varepsilon > 0$. Then y must achieve a new running minimum of some negative value at t, and this implies $z(t) = 0$.

For uniqueness, suppose that $\tilde{a}$ and $\tilde{z}$ are another pair of functions satisfying the properties above. Then $a - \tilde{a} = z - \tilde{z}$ is a process of bounded variation, and the integration by parts formula yields:

$$\begin{aligned}
0 \leq (z-\tilde{z})^2(t) &= 2\int_0^t (z(s) - \tilde{z}(s)) d(a-\tilde{a})(s) \\
&= 2\int_0^t z(s) da(s) - 2\int_0^t \tilde{z}(s) da(s) - 2\int_0^t z(s) d\tilde{a}(s) + 2\int_0^t \tilde{z}(s) d\tilde{a}(s) \\
&= -2\int_0^t \tilde{z}(s) da(s) - 2\int_0^t z(s) d\tilde{a}(s) \leq 0.
\end{aligned}$$

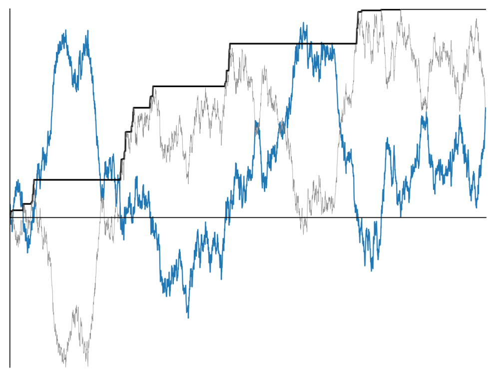
In the last line, we used that $\text{supp}(da) \subseteq \{t \geq 0 : z(t) = 0\}$ and $\text{supp}(d\tilde{a}) \subseteq \{t \geq 0 : \tilde{z}(t) = 0\}$ made two of the integrals vanish, and also that $z, \tilde{z} \geq 0$ implied that the remaining two integrals were non-negative. This proves that we have $z(t) = \tilde{z}(t)$ for all $t \geq 0$, hence $a(t) = \tilde{a}(t)$ for all $t \geq 0$. □

Now consider applying the lemma to the identity $|B_t| = \beta_t + L_t$. Almost surely, L is non-decreasing, continuous, vanishing at zero, and has supported contained in $\{t \geq 0 : B_t = 0\}$, as we saw in the last section. The remaining hypotheses are true by inspection. In particular this implies that a formula for the local time is $L_t = \sup_{s\in[0,t]} \beta_s^- = \sup_{s\in[0,t]}(-\beta_s)$. This implies that $\{|B_t|\}_{t\geq 0}$ is adapted to

$\{\mathcal{F}_t^\beta\}_{t\geq 0}$, hence $\mathcal{F}_t^\beta \supseteq \mathcal{F}_t^{|B|}$ for all $t \geq 0$. Combining this with our result from earlier, we have come to the surprising conclusion that $\{\mathcal{F}_t^\beta\}_{t\geq 0} = \{\mathcal{F}_t^{|B|}\}_{t\geq 0}$.

We can also apply the deterministic lemma to derive a powerful identity in law: Starting from the Brownian motion β, we get $(|B_t|, L_t)$ via the deterministic construction $L_t = \sup_{s\in[0,t]}(-\beta_s)$ and $|B_t| = \beta_t + L_t$. If we apply the same deterministic construction to an arbitrary Brownian motion B on any probability space, we get $(S_t - B_t, S_t)$ via $S_t = \sup_{s\in[0,t]}(-B_s)$. Since β is just a Brownian motion, this implies that the laws of $(S_t - B_t, S_t)$ and $(|B_t|, L_t)$ are identical. This is called Levy's identity and it makes the local time $\{L_t\}$ into a much more intuitive object.

The following image can be used to visualize Levy's identity:



We can also use Levy's identity to deduce some concrete facts about local times of Brownian motion. For example, since S_t is the inverse of a stable subordinator of index $1/2$, this shows that the same is true for the law of L_t. For another example, note that we have $S_t \to \infty$ almost surely, hence $L_t \to \infty$ almost surely. By the strong Markov property, this implies that $L_t^a \to \infty$ holds almost surely for each $a \in \mathbb{R}$, a statement which can be interpreted as saying that, for any given point, the Brownian motion spends an unbounded amount of time near it.

4. Tanaka's SDE

There are tons of applications of the theory of local times, and here we focus on one idea because it completes a conversation that we started earlier. We previously noted the difference between existence of weak solutions and strong solutions to a given SDE. In particular, we saw that SDEs of the form $dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dB_t$ admit strong solutions when μ and σ are Lipschitz in their second argument.

A classical example of an SDE which admits weak solutions but not strong solutions is due to Tanaka. (In particular this also shows that the Lipschitz condition above cannot be completely removed.) Consider the following, called Tanaka's SDE:

$$\begin{cases} dX_t = \operatorname{sign}(X_t)dB_t \\ X_0 = 0. \end{cases} \tag{27}$$

Of course, the function sign is not even continuous, hence not Lipschitz.

We already saw that this SDE admits a weak solution: If $\{X_t\}_{t\geq 0}$ starts as a Brownian motion on some space $(\Omega, \mathcal{F}, \mathbb{P})$, then the process $\{B_t\}_{t\geq 0}$ defined via

$dB_t = \text{sign}(X_t)dX_t$ has $\langle B, B\rangle_t = \int_0^t (\text{sign}(X_s))^2 ds = t$ and is hence a Brownian motion. Moreover, $dB_t = \text{sign}(X_t)dX_t$ implies $dX_t = \text{sign}(X_t)dB_t$, so the pair (X, B) is a weak solution to the SDE.

To see that this SDE does not admit a strong solution, assume that on an arbitrary space $(\Omega, \mathcal{F}, \mathbb{P})$ there were a Brownian motion $\{B_t\}_{t\geq 0}$ with natural filtration $\{\mathcal{F}_t^B\}_{t\geq 0}$, and that some $\{\mathcal{F}_t^B\}_{t\geq 0}$-adapted process $\{X_t\}_{t\geq 0}$ were a solution to the SDE pathwise almost surely. By Tanaka's formula and the SDE, this implies

$$|X_t| = \int_0^t \text{sign}(X_s)dX_s + L_t = B_t + L_t \tag{28}$$

where $\{L_t\}_{t\geq 0}$ represents the local time of X at zero. But X is a local martingale, so we know that

$$L_t = \lim_{\varepsilon\to 0} \frac{1}{2\varepsilon} \int_0^t \mathbf{1}_{[0,\varepsilon)}(|X_s|) d\langle X, X\rangle_s \tag{29}$$

holds almost surely, hence B is $\{\mathcal{F}_t^{|X|}\}_{t\geq 0}$ adapted. Since X is is $\{\mathcal{F}_t^B\}_{t\geq 0}$-adapted by assumption, we have $\mathcal{F}_t^X \subseteq \mathcal{F}_t^{|X|}$ for all $t \geq 0$. But this cannot be true because, for example $\{X_1 > 0\}$ is measurable with respect to $\mathcal{F}_1^X$ but not $\mathcal{F}_1^{|X|}$. So the SDE cannot admit strong solutions.