TIME-CHANGES OF CONTINUOUS SEMIMARTINGALES

Contents

1. Introduction

1.1. **Motivation.** It's often a useful idea to be able to change the rate at which a stochastic process travels through its path, a construction we call applying a "time change". One reason for studying this type of construction is that it plays nicely with the definition of a semimartingale: time changes of semimartingales remain semimartingales.

Another reason for doing studying time-changes is a representation theorem which states that many continuous local martingales can be made into a Brownian motion by applying a standard time change related to its quadratic variation; through this construction, one can sometimes deduce results about general continuous semimartingales from more concrete results about Brownian motion.

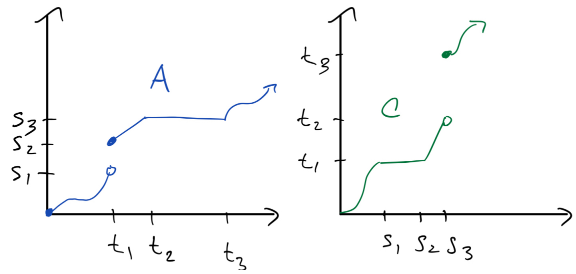
1.2. **Non-Random Time-Changes.** It makes sense to first study time-changes in a deterministic setting. If $A : \mathbb{R}_{\geq 0} \to \mathbb{R}$ is any non-decreasing, right-continuous function with $\lim_{t\to\infty} A_t = \infty$, then we can define the function $C : \mathbb{R}_{\geq 0} \to \mathbb{R}$ as

$$C_s = \inf\{t \geq 0 : A_t > s\}. \tag{1}$$

It follows that C is a non-decreasing, right-continuous function. Moreover, A and C are related via the above identity and also via

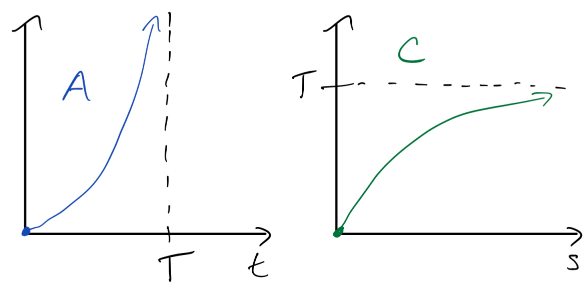
$$A_t = \inf\{s \geq 0 : C_s > t\}, \tag{2}$$

so it can be seen that A and C play dual roles, and each completely determines the other. By abuse of terminology, we'll call A and C "inverses" even though they are not always inveses in the exact sense. Also, for convenience of notation, we'll generally stick to the convention wherein we use t as the time variable in the "original time-line" and we'll use s as the time variable in the "transformed time-line". Consider the following image for a visualization of a pair of such functions:



Now we make some general remarks about the shape of such pairs of functions. First of all, note that jumps in A correspond to flat stretches in C, and flat stretches in A correspond to jumps in C. Observe also that A is continuous and strictly increasing if and only if C is continuous and strictly increasing, in which case the "inverses" A and C are honest-to-goodness inverses of each other.

Next we consider the behavior as time runs to infinity. For the sake of generality, we drop the condition that $\lim_{t\to\infty} A_t = \infty$ and the condition that A is defined on the entire domain $\mathbb{R}_{\geq 0}$. In fact, it is easy to see that if $\lim_{t\to\infty} A_t = S < \infty$, then C is only defined on the domain $[0, S)$. Dually, if A blows up in finite time and is hence only defined on the domain $[0, T)$ for $T < \infty$, then C satisfies the growth bound $\lim_{s\to\infty} C_s = T$. Consider the following image for an example of a pair of processes with this phenomenon



In the case above, that A blows up in finite time, we will extend our definition of A so that $A \equiv \infty$ on $[T, \infty]$. (Such an extension remains right-continuous.) Dually, we apply the same extension to C if C blows up in finite time. We say that a time-change C is *finite* if $C_s < \infty$ holds for all $s \geq 0$, or, equivalently, if $\lim_{t\to\infty} A_t = \infty$.

These remarks should not be seen as exceptions or pathologies; rather they should be seen as very important examples to include in the theory. Now we'll move onto the basic definitions and constructions in the random setting.

1.3. **Basic Definitions.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ denote a big probability space on which all of our random variables are defined, and let $\{\mathcal{F}_t\}$ denote a right-continuous filtration of $\mathcal{F}$.

Definition 1.1. A *time-change with respect to* $\{\mathcal{F}_t\}$ is a family $\{C_s\}$ of $\{\mathcal{F}_t\}$-stopping times whose sample paths are almost surely non-decreasing and right-continuous.

Given a time-change C and a stochastic process $\{X_t\}$, it makes sense to define the process $\{\hat{X}_s\}$ via $\hat{X}_s = X_{C_s}$. Of course, it would be silly to ask whether $\{\hat{X}_s\}$ is adapted, progressively measurable, predictable, etc. with respect to $\{\mathcal{F}_t\}$. Instead, it makes sense to consider the filtration $\{\hat{\mathcal{F}}_s\}$ defined via $\hat{\mathcal{F}}_s = \mathcal{F}_{C_s}$. Note that, from the right-continuity of $\{\mathcal{F}_t\}$ and $\{C_s\}$, we get the right continuity of $\{\hat{\mathcal{F}}_s\}$. Moreover, if $\{X_t\}$ is $\{\mathcal{F}_t\}$-progressively measurable, then $\{\hat{X}_s\}$ is $\{\hat{\mathcal{F}}_s\}$-adapted.

Now we should spend a moment solidifying the relationship between A and C in this random setting. Pay attention to the measurability statements here:

Lemma 1.2. If $\{A_t\}$ is an $\{\mathcal{F}_t\}$-adapted, non-decreasing, right-continuous process, then the random variables $\{C_s\}$ as defined by (1) are a time-change. Conversely, if $\{C_s\}$ is any time-change, then $\{A_t\}$ as defined by (2) is an $\{\mathcal{F}_t\}$-adapted, non-decreasing, right-continuous process.

Proof. That C is non-decreasing and right-continuous if and only if A is non-decreasing and right-continuous follows from the result in the non-random setting, which we do not prove here. To see that $\{A_t\}$ being $\{\mathcal{F}_t\}$-adapted implies that C_s is an $\{\mathcal{F}_t\}$-stopping time, note that we have the following identity for any $t_0 \geq 0$:

$$\{C_s < t_0\} = \bigcup_{\substack{t<t_0 \\ t\in\mathbb{Q}}} \{A_t > s\}. \tag{3}$$

Since $\{A_t > s\} \in \mathcal{F}_t \subseteq \mathcal{F}_{t_0}$, we have $\{C_s < t_0\} \in \mathcal{F}_{t_0}$. Since $\{\mathcal{F}_t\}$ is right-continuous, this implies $\{C_s \leq t_0\} \in \mathcal{F}_{t_0}$, hence C_s is an $\{\mathcal{F}_t\}$-stopping time.

Conversely, note that if every C_s is an $\{\mathcal{F}_t\}$-stopping time, then for any $s_0 \geq 0$ we have

$$\{A_t < s_0\} = \bigcup_{\substack{s<s_0 \\ s\in\mathbb{Q}}} \{C_s > t\}. \tag{4}$$

Now $\{C_s > t\} \in \mathcal{F}_t$, so $\{A_t < s_0\} \in \mathcal{F}_t$, hence A_t is $\mathcal{F}_t$-measurable. $\square$

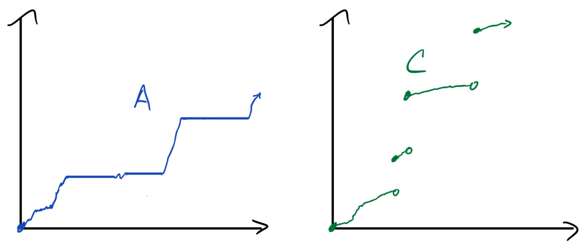
Now we consider the effect of time-changing on sample path properties of a process. Clearly, if $\{X_t\}$ is non-decreasing, then $\{\hat{X}_s\}$ is also non-decreasing. Also, if $\{X_t\}$ is right-continuous, then $\{\hat{X}_s\}$ is right-continuous. However, $\{\hat{X}_s\}$ can fail to be continuous when $\{X_t\}$ is continuous, due to the jumps in $\{C_s\}$. Hence it is useful to make the following definition which rules out this issue:

Definition 1.3. A process $\{X_t\}$ is called $\{C_s\}$-*continuous* if, almost surely, X is constant on the interval $[C_{t-}, C_t]$ for all $t \geq 0$.

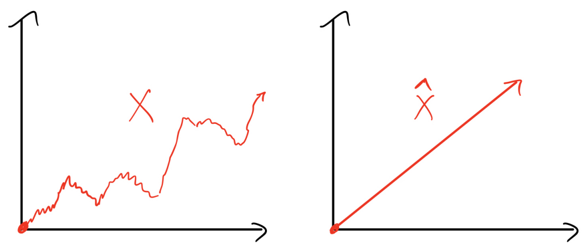
From here we see that if $\{X_t\}$ is continuous and $\{C_s\}$-continuous, then $\{\hat{X}_s\}$ is continuous. Another useful result related to the concept of $\{C_s\}$-continuity is that, for any semimartingale $\{Y_t\}$, the process Y has the same intervals of constancy as the process $\langle Y, Y\rangle$. (This was proved in an earlier section of R&Y.)

1.4. **Examples.** Let's focus on a few examples to illustrate these ideas. First, we note that, while deterministic time changes are certainly of interest, they are not so interesting from the probabilistic point of view. Examples abound, but we do not illustrate any here.

Now for a random-time change. Let $\{X_t\}$ be a standard Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration $\{\mathcal{F}_t\}$. Now set $A_t = \sup_{0 \le u \le t} X_u$, which is clearly $\{\mathcal{F}_t\}$-adapted. We know that A has many flat stretches, and it follows that the dual process C contains many jumps. For a generic Brownian sample path $\omega \in \Omega$, the sample paths $\{A_t(\omega)\}$ and $\{C_s(\omega)\}$ can be visualized as:

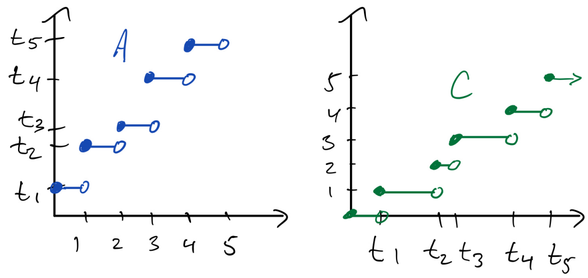


Now consider the time-changed process $\{\hat{X}_s\}$. For the same sample path ω as above, we plot $\{X_t(\omega)\}$ and $\{\hat{X}_s(\omega)\}$ and we get:

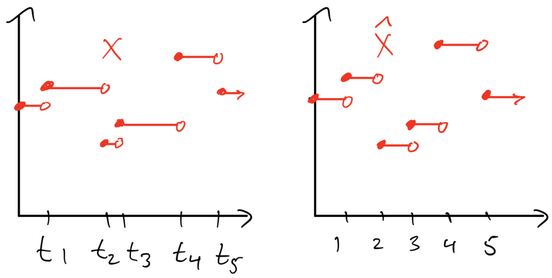


In other words, we claim that we have $\hat{X}_s = s$ for all $s \ge 0$. To see this, note that if $I \subseteq \mathbb{R}$ is an interval on which $A_t \equiv X_t$ holds, then C is exactly the inverse of A, which implies $\hat{X}_s \equiv s$ on $C^{-1}(I)$. On intervals J where A is flat, C experiences a jump, so $\hat{X}_s$ "skips" the interval J entirely. In other words, we have $C^{-1}(J) = \emptyset$, so $X_s \equiv s$ holds vacuously on J. This proves the result. (Note in this example that $\{X_t\}$ is not $\{C_s\}$-continuous.)

Now we consider a time change with important applications. Suppose that $\{Z_n\}_{n=1}^{\infty}$ is a discrete-time Markov chain with state space $\mathbb{R}$, and let $\lambda : \mathbb{R} \to (0, \infty)$ be any function. (Also assume that the transition kernel P for Z has $P(x, \{x\}) = 0$ so that there are no "unnoticeable" jumps.) Now let $\{N_t\}$ be the counting process which holds in state n for an independent exponential time of rate $\lambda(Z_n)$, then which jumps to $n+1$; let $\{X_t\}$ be the continuous-time Markov process defined as $X_t = Z_{N_t}$. From here, we define the process $\{A_t\}$ via $A_t = \inf\{u \ge 0 : \{X_t\}$ experiences $> t$ jumps in time $\le u\}$, and dually, the process $\{C_s\}$ has the property that C_s equals the number of jumps in $\{X_t\}$ in time $\le s$. Clearly each C_s is an $\{\mathcal{F}_t\}$-stopping time, so it follows that $\{A_t\}$ is $\{\mathcal{F}_t\}$-adapted. We can visualize the processes $\{C_s(\omega)\}_{s \ge 0}$ and $\{A_t(\omega)\}$ for a single $\omega \in \Omega$ as follows:



The utility of this example is that the time-changed process $\{\hat{X}_s\}$ has its jumps occurring exactly at times $s \in \mathbb{N}$, while the values of the process in between jumps are the same as those of $\{X_t\}$. Visualizing $\{X_t(\omega)\}$ and $\{\hat{X}_s(\omega)\}$, we see:



This specific time-change is referred to as *uniformization*, since it allows one to "recover" the discrete-time structure of a Markov chain which is embedded in a continuous-time Markov process. It is an important transformation in stochastic optimal control, since it allows one to transform some continuous-time problems into discrete-time problems for which direct analysis and computational approaches are both considerably simpler.

Lastly, we note that the concept of time-changing generalizes the concept of stopping. Indeed, if T is any $\{\mathcal{F}_t\}$-stopping time, then $C_s = s \wedge T$ is a time-change. Importantly, this is an example of a time-change C which has $\lim_{s\to\infty} C_s = T$, so its "inverse" A is not defined for all of $\mathbb{R}_{\geq 0}$.

2. General Theory

First we will establish that suitable time-changes of a continuous semimartingale yield a semimartingale. In particular, this will imply that a suitable time-change applied to a Brownian motion yield a continuous local martingale. Then we will study a few partial converses, which establish that a suitable continuous local martingale can be given a time change under which it becomes a Brownian motion.

2.1. Time-Changes of Semimartingales.

As before, let $(\Omega, \mathcal{F}, \mathbb{P})$ denote a probability space and $\{\mathcal{F}_t\}$ a right-continuous filtration of $\mathcal{F}$.

First we consider the effect of time-changing on processes of finite variation, and on their (Lebesgue-Stieljes) integrals:

Lemma 2.1. Let $\{C_s\}$ be a time change, and let $\{X_t\}$ be a continuous and $\{C_s\}$-continuous process of finite variation, both with respect to $\{\mathcal{F}_t\}$.

(1) The time-changed process $\{\hat{X}_s\}$ is continuous and has finite variation, with respect to the time-changed filtration $\{\hat{\mathcal{F}}_s\}$.

(2) If $\{H_t\}$ is $\{\mathcal{F}_t\}$-progressively measurable, then $\{\hat{H}_t\}$ is $\{\hat{\mathcal{F}}_s\}$-progressively measurable, and $\hat{H}\cdot\hat{X} = \widehat{H\cdot X}$.

Proof. (We drop the curly braces on our processes for notational convenience.) To prove (1), we see that that X being continuous is immediate from continuity and C-continuity. Then, if X has finite variation, then it can be written as $X = X_1 - X_2$ where X_1 and X_2 are non-decreasing. Now we have $\hat{X} = \hat{X}_1 - \hat{X}_2$, and $\hat{X}_1$ and $\hat{X}_2$ are both non-decreasing since this property is preserved under time-change. Therefore, $\hat{X}$ is of finite variation.

For (2), fix $s_0 \geq 0$, and define $f : \Omega \times [0, s_0] \to \Omega \times [0, C_{s_0}]$ via $f(\omega, s) = (\omega, C_s)$ which is clearly $(\mathcal{B}([0, s_0]) \otimes \hat{\mathcal{F}}_s; \mathcal{B}([0, C_{s_0}]) \otimes \mathcal{F}_{C_{s_0}})$-measurable. Then note that we can write $\hat{H} = H \circ f$, where H, in particular, is $(\mathcal{B}([0, C_{s_0}]) \otimes \hat{\mathcal{F}}_{C_{s_0}}; \mathcal{B}(\mathbb{R}))$-measurable. This shows that $\hat{H}$ is $(\mathcal{B}([0, s_0]) \otimes \mathcal{F}_{s_0}; \mathcal{B}(\mathbb{R}))$-measurable, hence $\hat{H}$ is $\{\hat{\mathcal{F}}_s\}$-progressively measurable. The second part follows by proving the following fact about deterministic time-changes for Lebesgue-Stieljes integrals:

$$\int_{C_0}^{C_s} H_u dX_u = \int_0^{s\wedge A_\infty} H_{C_u} dX_{C_u}. \tag{5}$$

To prove this, let dX and $d\hat{X}$ denote the Lebesgue-Stieljes measures for $\{X_t\}$ and $\{\hat{X}_s\}$, respectively. It suffices to prove that the pushforward measure $C_* d\hat{X}$ is equal to dX, since then the result will follow by applying the change-of-measure by C. For any interval $(t, t']$, we have $(dX)((t, t']) = X_{t'} - X_t$ by definition. Note that C being non-decreasing implies that $I = C^{-1}((t, t'])$ is an interval, it is clear that we only need to check a few cases to show $(C_* d\hat{X})((t, t']) = X_{t'} - X_t$: If $t \in (C_{s-}, C_s]$ for some $s \geq 0$, then the the left endpoint of I is closed at s, otherwise $t = C_s$ for some s so the left endopoint of I is open at s. If $t' \in (C_{s'-}, C_{s'}]$ for some $s' \geq 0$, then the the right endpoint of I is open at s', otherwise $t' = C_{s'}$ for some s' so the right endopoint of I is closed at s'. In any case we have, by the continuity and C-continuity of X, we have

$$\begin{aligned}(C_* d\hat{X})((t, t']) &= (d\hat{X})(C^{-1}((t, t'])) \\ &= X_{C_{s'}} - X_{C_s} \\ &= X_{t'} - X_t,\end{aligned}$$

so the result is proved. □

Next we consider the effect of time-changing on local martingales and on their (stochastic) integrals:

Lemma 2.2. Let $\{C_s\}$ be a time change which is finite almost surely (equivalently, that the inverse process has $\lim_{t\to\infty} A_t = \infty$) , and let $\{X_t\}$ be a continuous and $\{C_s\}$-continuous local martingale, both with respect to $\{\mathcal{F}_t\}$.

(1) The time-changed process $\{\hat{X}_s\}$ is a continuous local martingale with respect to $\{\hat{\mathcal{F}}_s\}$, and we have $\langle \hat{X}, \hat{X}\rangle = \widehat{\langle X, X\rangle}$.

(2) If $\{H_t\}$ is $\{\mathcal{F}_t\}$-progressively measurable and has $\int_0^t H_u^2 d\langle X, X\rangle_u < \infty$ almost surely for all $t \geq 0$, then we have $\int_0^s \hat{H}_u^2 d\langle \hat{X}, \hat{X}\rangle_u < \infty$ almost surely for all $s \geq 0$, and $\hat{H}\cdot\hat{X} = \widehat{H\cdot X}$.

Proof. For (1), let T be an $\{\mathcal{F}_t\}$-stopping time, and define the time-changed stopping time $S = \inf\{s \geq 0 : C_s \geq T\}$. We have, for any $s_0 \geq 0$, the identity

$$\{S < s_0\} = \bigcup_{\substack{s<s_0 \\ s\in\mathbb{Q}}} \{C_s \geq T\} \tag{6}$$

and that $\{C_s \geq T\}$ is $\hat{\mathcal{F}}_s$ measurable. Then $\hat{\mathcal{F}}_s \subseteq \hat{\mathcal{F}}_{s_0}$ implies $\{S < s_0\} \in \hat{\mathcal{F}}_{s_0}$, so the right-continuity of $\{\hat{\mathcal{F}}_s\}$ implies that S is an $\{\hat{\mathcal{F}}_s\}$-stopping time.

(For a moment we're going to switch to paranthetical notation instead of subscript notation, because iterated subscripts can get a little hard to read.) By definition we have $\hat{X}^S(s) = \hat{X}(s \wedge S) = X(C(s \wedge S))$. We would now like to use $X(C(s \wedge S)) = X(C(s) \wedge T)$ in order to apply some martingale properties of X to $\hat{X}$, but the fact that C may experience a jump at S necessitates an extra step. To do this, note that

$$A_T = \inf\{s \geq 0 : C_s > T\} \geq \inf\{s \geq 0 : C_s \geq T\} = S, \tag{7}$$

and also that $A_T \geq S$ implies $C_{S-} = \inf\{t \geq 0 : A_t \geq S\} \leq T$. Moreover, we have $C_S \geq T$, hence $T \in [C_{S-}, C_S]$. Therefore, the C-continuity of X implies that we have $\hat{X}^S(s) = X(C(s) \wedge T)$. This proves that, if X^T is a martingale with respect to $\{\mathcal{F}_t\}$, then $\hat{X}^S$ is a martingale with respect to $\{\hat{\mathcal{F}}_s\}$. So, if $\{T_n\}_{n=1}^{\infty}$ is a sequence of $\{\mathcal{F}_t\}$-stopping times such that $T_n \to \infty$ holds almost surely and such that X^{T_n} is a martingale for each n, then $\{S_n\}_{n=1}^{\infty}$ defined as $S_n = \inf\{s \geq 0 : C_s \geq T_n\}$ is a sequence of $\{\hat{\mathcal{F}}_s\}$-stopping times such that $\hat{X}^{S_n}$ is a martingale for each n. Moreover, since $\{C_s\}$ is finite almost surely and $T_n \to \infty$ holds almost surely, it follows that we have $S_n \to \infty$ almost surely. (Of course, if $\lim_{s\to\infty} C_s < \infty$, then the S_n may be an eventually constant sequence at ∞.) Therefore, $\{\hat{X}_s\}$ is a local martingale.

To see that $\langle \hat{X}, \hat{X} \rangle = \widehat{\langle X, X \rangle}$, note first that $\hat{X}^2 - \langle \hat{X}, \hat{X} \rangle$ is a local martingale by definition. Also, $X^2 - \langle X, X \rangle$ is a local martingale, and, since $\langle X, X \rangle$ has the same intervals of constancy as X, the C-continuity of X implies that $X^2 - \langle X, X \rangle$ is a C-continuous local martingale. Hence, (1) of this lemma (along with basic properties of the $\hat{}$ operation) implies that $\hat{X}^2 - \widehat{\langle X, X \rangle}$ is a local martingale. By uniqueness of the bracket, this forces $\langle \hat{X}, \hat{X} \rangle = \widehat{\langle X, X \rangle}$.

For (2), note that the first part follows from part (2) of the previous lemma, since $\langle X, X \rangle$ is an adapted process of finite variation, and since $\widehat{\langle X, X \rangle} = \langle \hat{X}, \hat{X} \rangle$. For the second part, it suffices to show that the quadratic variation of $\widehat{H \cdot X} - \hat{H} \cdot \hat{X}$ is identically zero. Expanding the quadratic variation via linearity we get

$$\begin{aligned} &\langle \widehat{H \cdot X} - \hat{H} \cdot \hat{X}, \widehat{H \cdot X} - \hat{H} \cdot \hat{X} \rangle \\ &\qquad = \langle \widehat{H \cdot X}, \widehat{H \cdot X} \rangle - 2\langle \widehat{H \cdot X}, \hat{H} \cdot \hat{X} \rangle + \langle \hat{H} \cdot \hat{X}, \hat{H} \cdot \hat{X} \rangle. \end{aligned}$$

Now we use some basic facts about stochastic integration, the result (2) of the previous lemma, and the result (1) of this lemma, to simplify these terms. In particular, we get

$$\begin{aligned}\langle \widehat{H\cdot X}, \widehat{H\cdot X}\rangle &= (\langle H\cdot X, H\cdot X\rangle)\hat{} \\ &= (H^2\cdot\langle X,X\rangle)\hat{} \\ &= \hat{H}^2\cdot\widehat{\langle X,X\rangle} \\ &= \hat{H}^2\cdot\langle\hat{X},\hat{X}\rangle,\end{aligned}$$

and

$$\begin{aligned}\langle \widehat{H\cdot X}, \hat{H}\cdot\hat{X}\rangle &= \hat{H}\cdot\langle\widehat{H\cdot X},\hat{X}\rangle \\ &= \hat{H}\cdot(\langle H\cdot X, X\rangle)\hat{} \\ &= \hat{H}\cdot(H\cdot\langle X,X\rangle)\hat{} \\ &= \hat{H}^2\cdot\widehat{\langle X,X\rangle} \\ &= \hat{H}^2\cdot\langle\hat{X},\hat{X}\rangle,\end{aligned}$$

and, of course $\langle\hat{H}\cdot\hat{X},\hat{H}\cdot\hat{X}\rangle = \hat{H}^2\cdot\langle\hat{X},\hat{X}\rangle$. Therefore, we have $\langle\widehat{H\cdot X}-\hat{H}\cdot\hat{X}, \widehat{H\cdot X}-\hat{H}\cdot\hat{X}\rangle = 0$, so $\hat{H}\cdot\hat{X} = \widehat{H\cdot X}$ holds. □

Combining these results we get the following important characterization of time-changes of semimartingales:

Corollary 2.3. Let $\{C_s\}$ be a time change which is finite almost surely, and let $\{X_t\}$ be a continuous and $\{C_s\}$-continuous semimartingale, both with respect to a right-continuous filtration $\{\mathcal{F}_t\}$.

(1) The time-changed process $\{\hat{X}_s\}$ is a continuous semimartingale with respect to the time-changed filtration $\{\hat{\mathcal{F}}_s\}$, and we have $\langle\hat{X},\hat{X}\rangle = \widehat{\langle X,X\rangle}$.

(2) If $\{H_t\}$ is $\{\mathcal{F}_t\}$-progressively measurable and has $\int_0^t H_u^2 d\langle X,X\rangle_u < \infty$ almost surely for all $t\geq 0$, then we have $\int_0^s \hat{H}_u^2 d\langle\hat{X},\hat{X}\rangle_u < \infty$ almost surely for all $s\geq 0$, and $\hat{H}\cdot\hat{X} = \widehat{H\cdot X}$.

Proof. Write $X = M + A$ for M a local martingale and A a continuous adapted process of finite variation. (The dual process of C does not appear in this proof, so there is no danger in letting A denote the finite-variation piece of X.) Since $\langle X,X\rangle = \langle M,M\rangle$ and X is C-continuous, we have that M is C-continuous, hence A is also C-continuous. Now (1) follows from part (1) of each of the respective lemmas above, and (2) follows from part (2) of each of the respective lemmas above. □

2.2. **Dambis-Dubins-Schwarz Theorem.** In this subsection we study a partial converse to the results of the last subsection. It shows that many processes admit a time change under which they become a Brownian motion.

Theorem 2.4 (Dambis, Dubins-Schwarz)**.** Suppose $\{X_t\}$ is a continuous local martingale with respect to $\{\mathcal{F}_t\}$, satisfying $X_0 = 0$ and $\lim_{t\to\infty}\langle X,X\rangle_t = \infty$ both almost surely. The time-change

$$C_s = \inf\{t\geq 0 : \langle X,X\rangle_t > s\} \tag{8}$$

has the property that that $\{\hat{X}_s\}$ is an $\{\hat{\mathcal{F}}_s\}$-Brownian motion, and also that $\{X_t\}$ can be written as $X_t = \hat{X}_{\langle X,X\rangle_t}$. We say that $\{\hat{X}_s\}$ is *the DDS Brownian motion of* $\{X_t\}$.

Proof. That $\{C_s\}$ is a time-change follows from the fact that its "inverse" $\{\langle X,X\rangle_t\}$ is non-decreasing, continuous, and $\{\mathcal{F}_t\}$-adapted. Since $\langle X,X\rangle_t \to \infty$ holds almost surely, we also have that $\{C_s\}$ is finite almost surely. Moreover, recall that C has jumps exactly where $\langle X,X\rangle$ has flat stretches, and that $\langle X,X\rangle$ and X have the same flat stretches. This shows that X is C-continuous, so the lemma above applies, and it shows that $\{\hat{X}_s\}$ is a local martingale with $\langle \hat{X},\hat{X}\rangle_s = \langle X,X\rangle_{C_s} = s$. So by Lévy's characterization, $\{\hat{X}_s\}$ is an $\{\hat{\mathcal{F}}_s\}$-Brownian motion.

For the second part, note that $\langle X,X\rangle$ is continuous so it has no jumps. However, the flat stretches of $\langle X,X\rangle$ correspond to jumps in C. But since X is constant wherever $\langle X,X\rangle$ is constant, it follows that we have $X_t = \hat{X}_{\langle X,X\rangle_t}$. □

The hypothesis that $\langle X,X\rangle_t \to \infty$ holds almost surely can be weakened at the cost of including some additional randomness in the underlying probability space. Specifically, by an *enlargement* of a filtered probability space $(\Omega,\mathcal{F},\{\mathcal{F}_t\},\mathbb{P})$ we mean another filtered probability space $(\tilde{\Omega},\tilde{\mathcal{F}},\{\tilde{\mathcal{F}}_t\},\tilde{\mathbb{P}})$ and a map $\pi : \tilde{\Omega} \to \Omega$ satisfying $\pi^{-1}(\mathcal{F}_t) \subseteq \tilde{\mathcal{F}}_t$ and $\pi_*\tilde{\mathbb{P}} = \mathbb{P}$ on $\mathcal{F}_t$ for all $t \geq 0$.

Also, we will need a previous result from earlier in R&Y which we now state but do not prove: If X is continuous local martingale, then $\lim_{t\to\infty} X_t$ exists almost surely on the set $\{\lim_{t\to\infty}\langle X,X\rangle < \infty\}$. Now we can state a generalization of the DDS theorem as follows:

Theorem 2.5. Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space with $\{\mathcal{F}_t\}$ a right-continuous filtration of $\mathcal{F}$, and let $\{X_t\}$ be a continuous local martingale with respect to $\{\mathcal{F}_t\}$. Also let $\{C_s\}$ be the time-change as in the DDS theorem. Then, there exists an enlargement $(\tilde{\Omega},\tilde{\mathcal{F}},\{\tilde{\mathcal{F}}_s\},\tilde{\mathbb{P}})$ of $(\Omega,\mathcal{F},\{\hat{\mathcal{F}}_s\},\mathbb{P})$ and an $\{\tilde{\mathcal{F}}_s\}$-Brownian motion $\{\tilde{B}_s\}$ independent of $\{X_t\}$ such that the process $\{B_s\}$ defined via

$$B_s = \begin{cases} \hat{X}_s & \text{if } s < \langle X,X\rangle_\infty \\ X_\infty + \tilde{B}_{s-\langle X,X\rangle_\infty} & \text{if } s \geq \langle X,X\rangle_\infty \end{cases} \tag{9}$$

is an $\{\tilde{\mathcal{F}}_s\}$-Brownian motion.

Proof. We use the natural construction via product spaces to create our enlargement. Let $(\Omega',\mathcal{F}',\mathbb{P}')$ denote a probability space on which a Brownian motion $\{B'_s\}$ is defined, and let $\{\tilde{\mathcal{F}}_s\}$ denote its natural filtration. Now set $\tilde{\Omega} = \Omega \times \Omega'$, define the filtration $\{\tilde{\mathcal{F}}_s\}$ via $\tilde{\mathcal{F}}_s = \hat{\mathcal{F}}_s \otimes \mathcal{F}'_s$, and define the measure $\tilde{\mathbb{P}} = \mathbb{P} \times \mathbb{P}'$. (Also, we should set $\tilde{\mathcal{F}} = \sigma(\bigcup_{s\geq 0}\tilde{\mathcal{F}}_s)$ as the ambient σ-algebra.) Now the process $\{\tilde{B}_s\}$ defined as $\tilde{B}_s(\omega,\omega') = B'_s(\omega')$ is an $\{\tilde{\mathcal{F}}_s\}$-Brownian motion.

Now note that the process $\{B_s\}$ can be written as

$$B_s = \hat{X}_s + \int_0^s \mathbf{1}\{u > \langle X,X\rangle_\infty\}\,d\tilde{B}_u. \tag{10}$$

This can be seen as follows: If $s < \langle X,X\rangle_\infty$, then $u < \langle X,X\rangle_\infty$ for all $u \leq s$, so the integral term is zero. If $s \geq \langle X,X\rangle_\infty$, then $C_s = \infty$ so $\hat{X}_s = X_\infty$, which exists almost surely because $\langle X,X\rangle_\infty < \infty$.

Since $\hat{X}_s$ and the stochastic integral term are both local martingales, it follows that $\{B_s\}$ is a local martingale. Moreover, some basic facts about stochastic integration show that its quadratic variation is:

$$\langle B, B\rangle_s = \langle \hat{X}, \hat{X}\rangle_s + \int_0^s \mathbf{1}\{u > \langle X, X\rangle_\infty\} du. \tag{11}$$

where the cross-term vanished because X and $\tilde{B}$ are independent. Now observe that this formula implies $\langle B, B\rangle_s = s$, which can be seen as follows: If $s < \langle X, X\rangle_\infty$, then the integral term vanishes and we have $\langle B, B\rangle_s = \langle \hat{X}, \hat{X}\rangle_s = s$, as in the proof of the original DDS theorem. Otherwise $s \geq \langle X, X\rangle_\infty$ and the integral term is equal to $s - \langle X, X\rangle_\infty \geq 0$. Adding this to the first term gives $\langle B, B\rangle_s = s$ as needed. So, by Lévy's characterization, the process $\{B_s\}$ is an $\{\tilde{\mathcal{F}}_s\}$-Brownian motion. □

2.3. **Knight's Theorem.** In this next subsection we address generalizing the DDS theorem to continuous local martingales in $\mathbb{R}^d$ for $d \geq 2$. (In many cases we particularly want to understand the $d = 2$ case.) To get started, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a right-continuous filtration $\{\mathcal{F}_t\}$. Let $\{X_t\}$ be an $\{\mathcal{F}_t\}$-adapted process taking values in $\mathbb{R}^d$, and, for each $1 \leq k \leq d$, let X_t^k denote the kth coordinate of X_t. Recall that, we say that $\{X_t\}$ is a local martingale with respect to $\{\mathcal{F}_t\}$ if $\{X_t^k\}$ is a local martingale with respect to $\{\mathcal{F}_t\}$ for each k.

The next result shows that, if $\{X_t\}$ is a local martingale in $\mathbb{R}^d$ with respect to $\{\mathcal{F}_t\}$, then there exists a time change for each coordinate such that the resulting process is a d-dimensional Brownian motion.

Theorem 2.6 (Knight)**.** Let $\{X_t\}$ be a local martingale in $\mathbb{R}^d$ with respect to $\{\mathcal{F}_t\}$ such that $X_0 = 0$ and such that $\lim_{t\to\infty}\langle X^k, X^k\rangle_t = \infty$ and $\langle X^k, X^\ell\rangle_t = 0$ hold almost surely for all $1 \leq k \neq \ell \leq d$. Then define, for each k:

$$C_s^k = \inf\{t \geq 0 : \langle X^k, X^k\rangle_t > s\}. \tag{12}$$

It follows that the process $\{\hat{X}_s\}$ defined via $\hat{X}_s^k = X_{C_s^k}^k$ is a d-dimensional Brownian motion (with respect to its natural filtration).

Proof. By the DDS theorem, each coordinate $\{\hat{X}_t^k\}$ is a Brownian motion. So, it only remains to show that the coordinates are independent. To do this, let $0 \leq t_0 < t_1 < \cdots < t_p$ and $\lambda_1, \ldots \lambda_p \in \mathbb{R}^d$ be arbitrary, and define $f_k = \sum_{j=1}^p \lambda_j^{(k)} \mathbf{1}_{(t_{j-1}, t_j]}$. Now set

$$Y_t = \sum_{k=1}^d \int_0^t f_k(s) d\hat{X}_s^k, \tag{13}$$

and note that our previous results and the assumption that $\langle X^k, X^\ell\rangle_t = 0$ imply that the quadratic variation of Y is equal to

$$\langle Y, Y\rangle_t = \sum_{k=1}^d \int_0^t f_k^2(s) ds. \tag{14}$$

Now recall by Ito's formula that $I_t = \exp(iY_t + \frac{1}{2}\langle Y, Y\rangle_t)$ is a complex local martingale, but $|Y_t| \leq \exp(\frac{1}{2}\sum_{k=1} ||f_k||_{L^2}^2)$ implies that it is bounded and hence a martingale. This implies that we have $\mathbb{E}[Y_t] = \mathbb{E}[Y_0] = 1$, hence

$$\mathbb{E}\left[\exp\left(i\sum_{k=1}^{d}\sum_{j=1}^{p}\lambda_j^{(k)}(\hat{X}_{t_j} - \hat{X}_{t_{j-1}})\right)\right] = \mathbb{E}\left[\exp\left(-\frac{1}{2}\sum_{k=1}^{d}\sum_{j=1}^{p}\left(\lambda_j^{(k)}\right)^2(t_j - t_{j-1})\right)\right].$$

The right side is the characteristic function for a multivariate Gaussian with independent coordinates, which finishes the proof. □

A few remarks are now necessary. First, as expected by analogy with the case of the DDS theorem, there is a generalization of Knight's theorem which allows the coordinate-wise quadratic variations to be bounded, provided that we englarge the underlying probability space. We do not state or prove this generalized version here, but details can be found in R&Y.

Next we remark that, since different time-changes may be required for each coordinate, there is no time-changed filtration to which we can immediately say that $\{B_s\}$ is adapted. However, if the coordinate-wise quadratic variations are all identical, then inspection of the proof reveals that such a time-changed filtration does exist.

3. Applications

In this last section we turn to an application of these ideas of time-changes that we have just studied. The approach for all of these examples is more or less the same: If there is a known property of Brownian motion which is preserved under time-change, then we can "bootstrap" the same result to any continuous local martingale by the DDS theorem.

As a first example, we prove a general version of the law of the iterated logarithm. Recall that for a Brownian motion $\{B_t\}$ we have, almost surely:

$$\limsup_{t\to\infty}\frac{B_t}{\sqrt{2t\log\log t}} = 1 \qquad \text{and} \qquad \liminf_{t\to\infty}\frac{B_t}{\sqrt{2t\log\log t}} = -1. \tag{15}$$

Now let $\{X_t\}$ be a continuous local martingale. Let $\{\hat{X}_s\}$ be the DDS Brownian motion of $\{X_t\}$, which we may need to enlarge the probability space to acquire. Recall from the DDS theorem that we can write $\hat{X}_{\langle X,X\rangle_t} = X_t$. Therefore, on the event $\{\langle X, X\rangle_\infty = \infty\}$, we have the following almost surely:

$$\limsup_{t\to\infty}\frac{X_t}{\sqrt{2\langle X,X\rangle_t\log\log\langle X,X\rangle_t}} = 1$$
$$\text{and} \qquad \liminf_{t\to\infty}\frac{X_t}{\sqrt{2\langle X,X\rangle_t\log\log\langle X,X\rangle_t}} = -1.$$

As in the case of Brownian motion, this result characterizes the sharp denominator for which a limit almost surely does not exist.

For another application, we prove the Burkholder-Davis-Gundy inequalities by assuming that the result is known for Brownian motion. (Proving the result for Brownian motion is not trivial, but it is at least easier than the general case.) In other words, assume that, for any $p \in (0, \infty)$, there exists constants $c_p, C_p > 0$ such that for any stopping time T we have

$$c_p \mathbb{E}\left[T^{\frac{p}{2}}\right] \leq \mathbb{E}\left[\left(\sup_{0 \leq t \leq T} B_t\right)^p\right] \leq C_p \mathbb{E}\left[T^{\frac{p}{2}}\right]. \tag{16}$$

Now suppose that $\{X_t\}$ is a continuous local martingale and $\{\hat{X}_s\}$ is its DDS Brownian motion. Using $\hat{X}_{\langle X,X\rangle_t} = X_t$ and the stopping time $T = \lim_{t\to\infty}\langle X, X\rangle_t$, this implies

$$c_p \mathbb{E}\left[\left(\lim_{t\to\infty}\langle X, X\rangle_t\right)^{\frac{p}{2}}\right] \leq \mathbb{E}\left[\left(\sup_{t\geq 0} X_t\right)^p\right] \leq C_p \mathbb{E}\left[\left(\lim_{t\to\infty}\langle X, X\rangle_t\right)^{\frac{p}{2}}\right], \tag{17}$$

where $c_p, C_p > 0$ are are before.

Another large collection of applications is in the circle of ideas around the conformal invariance (up to time-change) of planar Brownian motion. This is the content of the next section of R&Y, so we won't go through the details today.