On Exchangeable Random Colororings of Infinite, Complete, Uniform Hypergraphs

A. Q. Jaffe
STAT C206 / MATH C223B

May 22, 2020

Introduction

Notation

- Write $[n] = \{1, 2, \ldots n\}$, and also write $\binom{S}{k}$ for the set of all size-k subsets of a set S, as well as $\binom{S}{\leq k} = \bigcup_{i=0}^{k} \binom{S}{i}$. Also let $\{*\}$ denote any one-point set, and let $|\cdot|$ denote the cardinality of a set.
- We will use $(\Omega, \mathcal{F}, \mathbb{P})$ to denote a big probability space on which all of our random variables are defined
- Let $\mathcal{B}(\mathcal{X})$ denote the Borel σ-algebra of a topological space $\mathcal{X}$, and let $\mathcal{X}^T$ be always endowed with the product topology.
- A random element of a product space K^T can equivalently be viewed as the canonical coordinate process, $\{\pi_t\}_{t \in T}$. This will allows us to interchange between discussing its law μ and its probabilities under $\mathbb{P}$.
- If $\psi : T \to T$ is a bijection and K is any set, then there is a natural induced bijection on the product space K^T defined via $\tau^{\psi}(\{\pi_t\}_{t \in T}) = \{\pi_{\psi(t)}\}_{t \in T}$. This extends naturally to a bijection on spaces $K_1^T \times K_2^{\binom{T}{2}} \times \cdots \times K_k^{\binom{T}{k}}$ when $K_1, \ldots K_k$ are any sets.

Definitions

- For a (usually infinite) vertex set S, the *complete k-uniform hypergraph on S* is $\binom{S}{k}$, and elements of $\binom{S}{k}$ are called *hyperedges*. (a disconnected graph is $k = 1$, standard graphs are $k = 2$, etc.)
- For K a standard Borel space, a *K-coloring* of a k-uniform hypergraph is a map $H : \binom{S}{k} \to K$, or equivalently a collection of colors $\{\pi_u\}_{u \in \binom{S}{k}}$.
- A *random K-coloring of* $\binom{S}{k}$ is a probability measure on $K^{\binom{S}{k}}$, or equivalently the canonical coordinate process $\{\pi_u\}_{u \in \binom{S}{k}}$.
- We say that a random k-coloring of $\binom{S}{k}$ is *invariant* under a permutation $\psi : S \to S$ if $\{\pi_{\psi(u)}\}_{u \in \binom{S}{k}}$ has the same law as $\{\pi_u\}_{u \in \binom{S}{k}}$, or equivalently, if $\mu \circ (\tau^{\psi})^{-1} = \mu$. A random K-coloring of S which is invariant under all finite permutation is called *exchangeable*.
- **Motivating question:** Can we describe all of the exchangeable random K-colorings of $\binom{S}{k}$?

Examples

- It's quite easy to come up with lots of different exchangeable random K-colorings of $\binom{S}{k}$:
  - *de Finetti construction.* Fix a mixture of distributions over K; first sample the distribution, then sample each hyperedge conditionally independently from the chosen distribution.
  - *Vertex-sampling construction.* Let H be some fixed K-coloring of a k-uniform hypergraph on n vertices. Then, let the vertices $s \in S$ be sampled as iid from the uniform distribution on $\{1, \dots n\}$. Now, define the color of the hyperedge $\{s_1, \dots s_k\}$ to be $H(s_1, \dots s_k)$.
- Both of these procedures give rise to exchangeable random K-colorings of an infinite k-uniform hypergraph, so any characterization of exchangeable colorings must include both examples above (and many many more).
- It turns out that, in order to capture all of the diversity of posssible colorings, we need to allow randomness to enter "at every level".

The Standard Recipe

- A collection of *ingredients* is a sequence of standard Borel spaces $Z_0, \dots Z_{k-1}, Z_k = K$, a probability measure μ_0 on Z_0, and a a collection of probability kernels $P_1, \dots P_k$ such that P_i is a kernel from $Z_0 \times Z_1^i \times Z_2^{\binom{i}{2}} \times \cdots \times Z_{i-1}^{\binom{i}{i-1}}$ to Z_i, such that each P_i is invariant under finite permutations of S.
- The *standard recipe* is the following recursive algorithm for sampling a K-coloring of a k-uniform hypergraph $\binom{S}{k}$:
  - Base Case: Sample a random $z_\emptyset \in Z_0$ according to μ_0.
  - Inductively for $2 \leq i \leq k$: For each hyperedge $u \in \binom{S}{i}$, sample $z_u \in Z_i$ according to the law

$$P_i\left(z_\emptyset, z_u, \{z_a\}_{a\in\binom{u}{2}}, , \cdots \{z_a\}_{a\in\binom{u}{i-1}}, \ \cdot\ \right). \tag{1}$$

  - Let $\{z_u\}_{u\in\binom{S}{k}}$ be the final coloring.

Structure Theorem

If we start with any collection of ingredients, then it is clear that the output of the standard recipe is an exchangeable random K-coloring of the K-uniform hypergraph $\binom{S}{k}$. In fact, our main result is that this is the only game in town:

Theorem (2.9 of [Aus08])

If μ is any exchangeable random K-coloring of a k-uniform hypergraph $\binom{S}{k}$, then there exists a collection of ingredients $(Z_0, \mu_0), (Z_1, P_1), \cdots (Z_{k-1}, P_{k-1}), (K, P_k)$ such that the output of the standard recipe with these ingredients is a random element of $K^{\binom{S}{k}}$ with law μ.

This result implies de Finetti's theorem, a structure theorem for exchangeable arrays, and more.

Outline

- Remainder of the talk has three parts:

($k = 1$) We'll introduce a clever proof of de Finetti's theorem in order to acquaint ourselves with the tools that we'll need for the general case.

($k = 2$) We'll prove the result for exchangeable edge-colorings of graphs. While this result will eventually be implied as a special case of the general result, it is a good setting to see our tools in action for a case that slightly more subtle than the easy de Finetti theorem.

($k \geq 2$) We'll move to an ostensibly harder statement, prove it by induction on k, and then conclude the structure theorem as a special case.

- I'll mostly be following the survey article [Aus08], in which the material of this presentation corresponds to subsections 3.1, 3.2, and 3.3. At times the survey is sparse on the details, so I'll also draw inspiration from [Ald85]. In either case, I'll be modifying their notation as necessary.
- A lot of the steps of the proof are best understood visually, so I've added some graphics to make things more intuitive.

Useful Results

The following results are some simple facts that will be used many times throughout the results in this presentation.

- If S is an infinite set, $V \subseteq S$ is infinite, and $S \setminus V$ is infinite, then there is a bijection between S and V.
- *Energy-Squeeze.* If $\mathcal{F} \subseteq \mathcal{G}$ are nested σ-algebras and $\mathbb{E}|\mathbb{E}[X|\mathcal{F}]|^2 = \mathbb{E}|\mathbb{E}[X|\mathcal{G}]|^2$, then $\mathbb{E}[X|\mathcal{G}]$ is $\mathcal{F}$-measurable.
- *Conditioning by Disintegration.* If X and Y are random elements of standard Borel spaces $\mathcal{X}$ and $\mathcal{Y}$ respectively, then there exists a kernel P from Y to X such that $\mathbb{E}[X|\sigma(Y)]$ is $\mathbb{P}$-almost surely distributed according to $P(Y, \cdot)$.

de Finetti's Theorem, $k = 1$

Statement

Theorem (de Finetti, 1937)

Let K be a standard Borel space and S a countable infinite index set. If μ is the law of an exchangeable random K-coloring of S, then there exists a standard Borel space Z_0, a Borel measure μ_0 on Z_0, and a kernel P_1 from Z_0 to K such that the output of the standard recipe with ingredients $(Z_0, \mu_0), (K, P_1)$ is a random element of K^S with distribution μ.

- We'll introduce a new proof of this, which will get us used to the ideas/constructions that we'll need for the more general results
- We will rely on S being infinite many many times throughout this proof and throughout the remaining proofs.
- Let's outline the key steps of the proof before proving them and then putting it all together.

Proof Outline

- **Set-Up:** Partition S into two infinite sets $S_1 \cup S_2$. Let T^V be the σ-algebra generated by $\{\pi_s\}_{s \in V}$ for any $V \subseteq S_2$.
- **Step 1.** Show that, for any $s \in S_1$, the conditional distribution of π_s given T^{S_2} is T^V-measurable.
- **Step 2.** Show that $\{\pi_s\}_{s \in S_1}$ are conditionally independent given T^{S_2}.
- **Step 3.** Show that $\{\pi_s\}_{s \in S_1}$ are conditionally identically distributed given T^{S_2}.

Proof of Step 1.

Note that we can write

$$\mathbb{E}\left|\mathbb{E}\left[f(\pi_s)\middle|T^V\right]\right|^2 = \sup_{\substack{F\subseteq V\\ F\,\text{finite}}} \mathbb{E}\left|\mathbb{E}\left[f(\pi_s)\middle|\sigma(\pi_s; s\in F)\right]\right|^2. \qquad (2)$$

However, the right side depends only on $|F|$, since for any F and F' of the same size we can swap them and the law of the conditional expectation does not change. Since V and S_2 both contain arbitrarily large subsets, this implies

$$\mathbb{E}\left|\mathbb{E}\left[f(\pi_s)\middle|T^V\right]\right|^2 = \mathbb{E}\left|\mathbb{E}\left[f(\pi_s)\middle|T^{S_2}\right]\right|^2. \qquad (3)$$

Since $T^V \subseteq T^{S_2}$, energy-squeezing implies that $\mathbb{E}[f(\pi_s)|T^{S_2}]$ is T_V-measurable. □

Proof of Step 2.

To show that these random variables are conditionally independent, it suffices to show that, for any bounded measurable functions $f_1, \dots f_r : K \to \mathbb{R}$ and any fixed $s_1, \dots s_r \in S_1$, we have

$$\mathbb{E}\left[f_1(\pi_{s_1}) \cdots f_r(\pi_{s_r}) \middle| T^{S_2}\right] = \mathbb{E}\left[f_1(\pi_{s_1}) \middle| T^{S_2}\right] \cdots \mathbb{E}\left[f_r(\pi_{s_r}) \middle| T^{S_2}\right]. \tag{4}$$

By induction on r, it suffices to show that we can pull out the $f_1(\pi_{s_1})$ term. That is, for μ the law of $\{\pi_s\}_{s \in S}$, it suffices to show that for any $A \in T^{S_2}$ we have

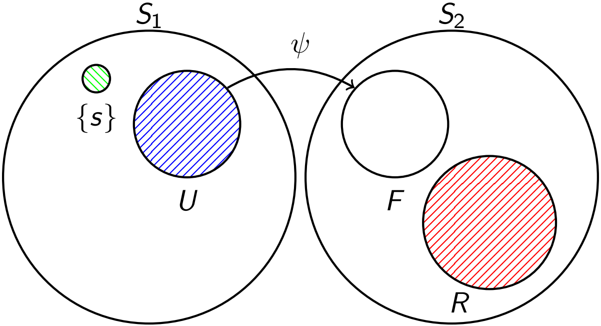
$$\int_A f_1(\pi_{s_1}) \cdots f_r(\pi_{s_r}) d\mu = \int_A \mathbb{E}\left[f_1(\pi_{s_1}) \middle| T^{S_2}\right] f_2(\pi_{s_2}) \cdots f_r(\pi_{s_r}) d\mu. \tag{5}$$

Proof of Step 3.

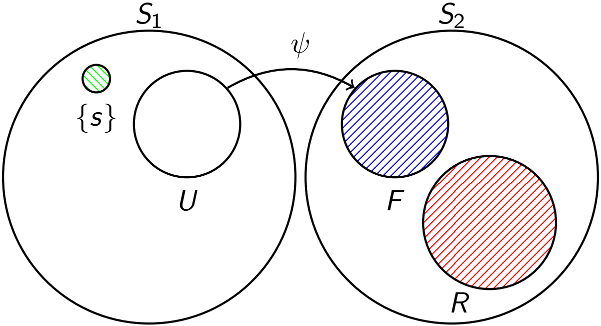
Now recall that for every $A \in \mathcal{B}(K^{S_2})$, we can get a sequence of cylinder sets $\{A_n\}$ in K^{S_2} with $A_n \uparrow A$ and $\mu(A \setminus A_n) \to 0$, so by dominated convergence we have $\int_{A_n} g d\mu \to \int_A g d\mu$ for all bounded measurable g.

Therefore, it suffices to prove (5) when A depends on only a finite set of coordinates $R \subseteq S_2$. To do this, let ψ be the (finite) permutation that swaps $U = \{s_2, \ldots s_r\}$ with some finite set $F \subseteq S_2 \setminus R$ and hence which leaves s_1 fixed.

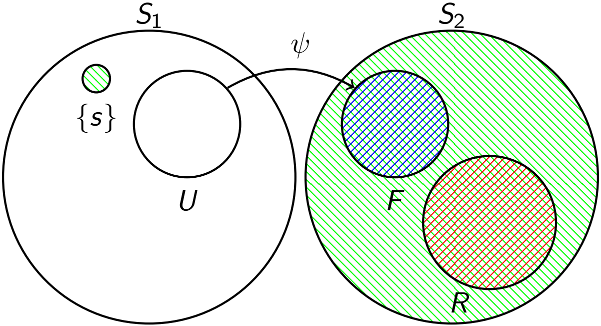
The next step is to apply exchangeability and the result of Step 1 many times to move through a large string of equalities. This is best explained through the following graphic:



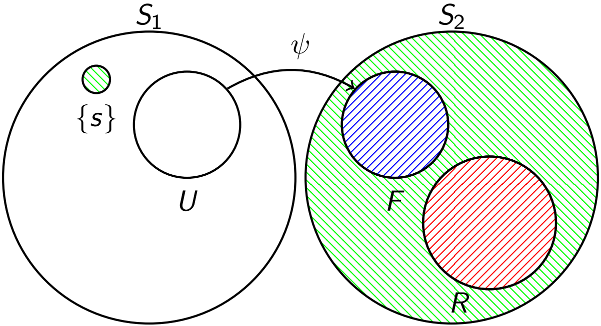
$$\int_{A} f_1(\pi_{s_1}) f_2(\pi_{s_2}) \cdots f_r(\pi_{s_r}) d\mu$$



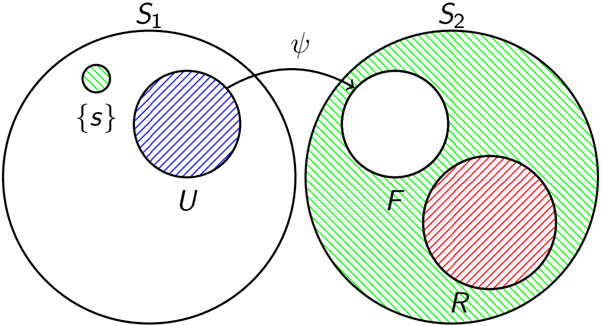
$$\int_A f_1(\pi_{s_1}) f_2(\pi_{\psi(s_2)}) \cdots f_r(\pi_{\psi(s_r)})\, d\mu$$



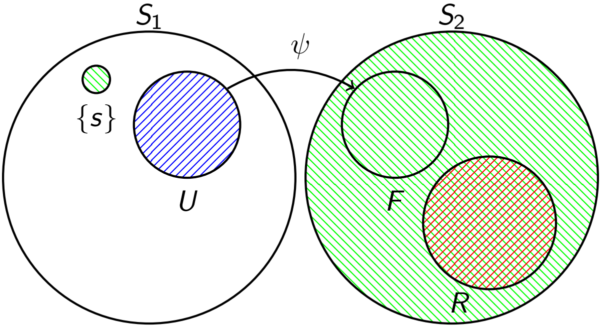
$$\int_A \mathbb{E}\left[f_1(\pi_{s_1}) \middle| T^{S_2}\right] f_2(\pi_{\psi(s_2)}) \cdots f_r(\pi_{\psi(s_r)}) d\mu$$



$$\int_A \mathbb{E}\left[f_1(\pi_{s_1}) \,\middle|\, T^{S_2\setminus(R\cup F)}\right] f_2(\pi_{\psi(s_2)})\cdots f_r(\pi_{\psi(s_r)})\,d\mu$$



$$\int_A \mathbb{E}\left[f_1(\pi_{s_1})\,\middle|\, T^{S_2\setminus(R\cup F)}\right] f_2(\pi_{s_2})\cdots f_r(\pi_{s_r})\,d\mu$$



$$\int_A \mathbb{E}\left[f_1(\pi_{s_1}) \middle| T^{S_2} \right] f_2(\pi_{s_2}) \cdots f_r(\pi_{s_r}) d\mu$$

Proof of Step 3.

In other words, we have:

$$\begin{aligned}
&\int_A f_1(\pi_{s_1}) f_2(\pi_{s_2}) \cdots f_r(\pi_{s_r}) d\mu \\
&\quad = \int_A f_1(\pi_{s_1}) f_2(\pi_{\psi(s_2)}) \cdots f_r(\pi_{\psi(s_r)}) d\mu \\
&\quad = \int_A \mathbb{E}\left[f_1(\pi_{s_1}) \middle| T^{S_2} \right] f_2(\pi_{\psi(s_2)}) \cdots f_r(\pi_{\psi(s_r)}) d\mu \\
&\quad = \int_A \mathbb{E}\left[f_1(\pi_{s_1}) \middle| T^{S_2 \setminus (R \cup F)} \right] f_2(\pi_{\psi(s_2)}) \cdots f_r(\pi_{\psi(s_r)}) d\mu \\
&\quad = \int_A \mathbb{E}\left[f_1(\pi_{s_1}) \middle| T^{S_2 \setminus (R \cup F)} \right] f_2(\pi_{s_2}) \cdots f_r(\pi_{s_r}) d\mu \\
&\quad = \int_A \mathbb{E}\left[f_1(\pi_{s_1}) \middle| T^{S_2} \right] f_2(\pi_{s_2}) \cdots f_r(\pi_{s_r}) d\mu,
\end{aligned}$$

which is exactly (5), as needed.

Proof of Step 3.

Take any $s_1, s_2 \in S_1$ and any bounded measurable $f : K \to \mathbb{R}$. Applying exchangeability to the permutation that swaps s_1 and s_2 and leaves all other vertices invariant, we get

$$\int_A f(\pi_{s_1}) d\mu = \int_A f(\pi_{s_2}) d\mu \tag{6}$$

for all $A \in T^{S_2}$. By the definition of conditional expectation, we have

$$\mathbb{E}\left[f(\pi_{s_1}) \middle| T^{S_2} \right] = \mathbb{E}\left[f(\pi_{s_2}) \middle| T^{S_2} \right]. \tag{7}$$

as needed. □

Finishing the Proof.

Consider the conditional distribution of $\{\pi_s\}_{s\in S_1}$ given T^{S_2}. By Steps 2 and 3, the random variables $\{\pi_s\}_{s\in S_1}$ are conditionally iid given T^{S_2}, so there exists a kernel P_1 from $Z_0 = K^{S_2}$ to K satisfying

$$\mu(A) = \int_{Z_0} P_1^{\otimes S_1}(z, A)\, d\mu_0(z) \tag{8}$$

for all $A \in \mathcal{B}(K^{S_1})$, where we have defined μ_0 to be the law of $\{\pi_s\}_{s\in S_2}$. This proves the result for the law of $\{\pi_s\}_{s\in S_1}$. But the laws of $\{\pi_s\}_{s\in S_1}$ and $\{\pi_s\}_{s\in S}$ are the same, so the theorem is proven for $\{\pi_s\}_{s\in S}$ with the choice (Z_0, μ_0) and (K, P_1) as above. □

Exchangeable Edge-Colorings, $k = 2$

Statement

Theorem

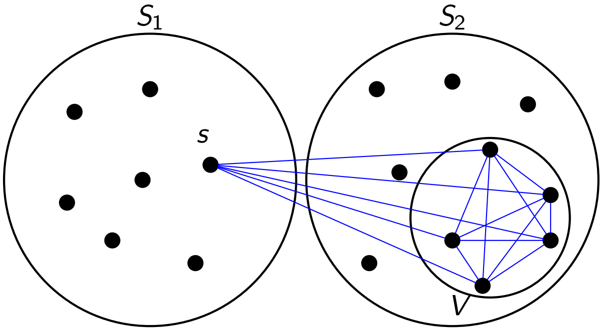
Let μ be the law of a random exchangeable K-coloring of $\binom{S}{2}$ where K is any standard Borel space. Then there exist ingredients $(Z_0, \mu_0), (Z_1, P_1), (K, P_2)$ such that the output of the standard recipe with these ingredients is a random element of $K^{\binom{S}{2}}$ with distribution μ.

- The idea of the proof is very similar to the $k = 1$ case, but the notation and indexing will be a little bit more confusing.
- We'll outline the key steps in the proof and then put it all together at the end

Proof Outline

- **Set-Up:** Partition S into two infinite sets $S_1 \cup S_2$. For any $s \in S_1$ and $V \subseteq S_2$, let $\pi_s^V = \{\pi_u : u \in \binom{\{s\} \cup V}{2}\}$. Also set $T^V = \sigma(\pi_s^V : s \in S_1)$ for $V \subseteq S_2$.
- **Step 1.** Show that, for any $e \in \binom{S_1}{2}$ and any infinite $V \subseteq S_2$, the conditional distribution of π_e given T^{S_2} is T^V-measurable.
- **Step 2.** Show that the random variables $\{\pi_e : e \in \binom{S_1}{2}\}$ are conditionally independent given T^{S_2}
- **Step 3.** Show that, for any distinct $s, t \in S_1$, the conditional distribution of $\pi_{\{s,t\}}$ given T^{S_2} is $\sigma(\pi_s^{S_2}, \pi_t^{S_2})$-measurable.
- **Step 4.** Show that $\{\pi_s^{S_2}\}_{s \in S_1}$ is exchangeable.

π_s^V



Proof of Step 1.

The proof is essentially identical to the proof for $k = 1$. Note that for any $e \in \binom{S_1}{2}$, we can write

$$\begin{aligned}&\mathbb{E}\left|\mathbb{E}\left[f(\pi_e)\middle| T^V\right]\right|^2 \\ &= \sup_{\substack{F \subseteq V \\ F \text{finite}}} \mathbb{E}\left|\mathbb{E}\left[f(\pi_e)\middle|\sigma\left(\pi_a; a \in \binom{e \cup F}{2} \setminus e\right)\right]\right|^2.\end{aligned}$$

By exchangeability, the right side depends only on $|F|$, since for any F and F' of the same size we can swap them and the law of the conditional expectation does not change. Since V and S_2 both contain arbitrarily large subsets, this implies

$$\mathbb{E}\left|\mathbb{E}\left[f(\pi_e)\middle| T^V\right]\right|^2 = \mathbb{E}\left|\mathbb{E}\left[f(\pi_e)\middle| T^{S_2}\right]\right|^2, \tag{9}$$

so energy-squeezing implies that $\mathbb{E}[f(\pi_e)|T^{S_2}]$ is T_V-measurable. □

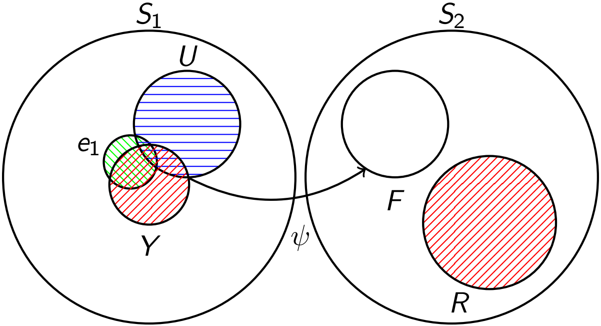
Proof of Step 2.

Take any bounded measurable functions $f_1, \dots f_r : K \to \mathbb{R}$ and any fixed $e_1, \dots e_r \in \binom{S_1}{2}$. As in the case of $k = 1$, it suffices (by induction on r) to show that we have the following for all $A \in T^{S_2}$:

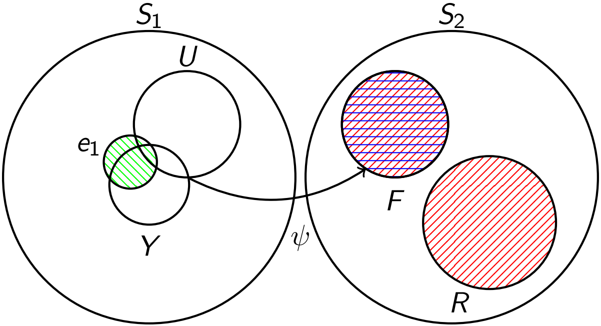
$$\int_A f_1(\pi_{e_1}) \cdots f_r(\pi_{e_r}) d\mu = \int_A \mathbb{E}\left[f_1(\pi_{e_1}) \middle| T^{S_2} \right] f_2(\pi_{e_2}) \cdots f_r(\pi_{e_r}) d\mu. \quad (10)$$

Again, it suffices to prove this in the case that A depends on only finitely many coordinates. So, we can assume that A is actually $\sigma(\pi_e; e \in \binom{\{y_j\} \cup R_j}{2}, 1 \leq j \leq m)$-measurable, where $y_j \in S_1$ and $R_j \subseteq S_2$ is finite for all $1 \leq j \leq m$. Then let $Y = \{y_1, \dots y_m\}$, $R = \bigcup_{i=1}^{m} R_i$, and $U = \bigcup_{i=2}^{m} e_i$. Let ψ be the (finite) permutation that swaps $(U \cup Y) \setminus e_1$ with some finite set of vertices $F \subseteq S_2 \setminus R$ (and hence which leaves the vertices in e_1 fixed).

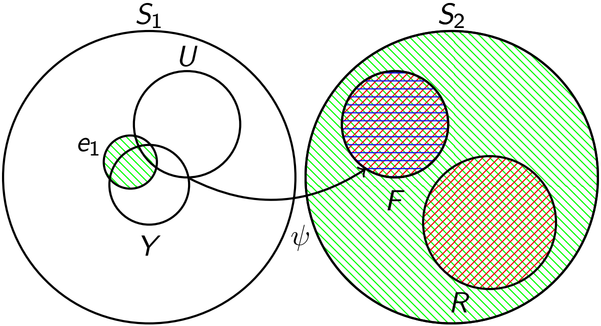
As before, let's do a graphical proof of the desired applications of Step 1 and exchangeability before showing the string equalities that it gives.



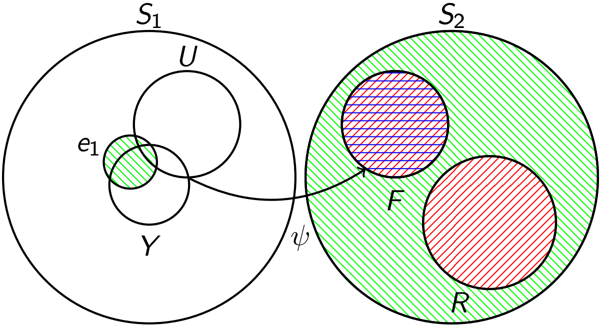
$$\int_{A} f_1(\pi_{e_1}) f_2(\pi_{e_2}) \cdots f_r(\pi_{e_r}) \, d\mu$$



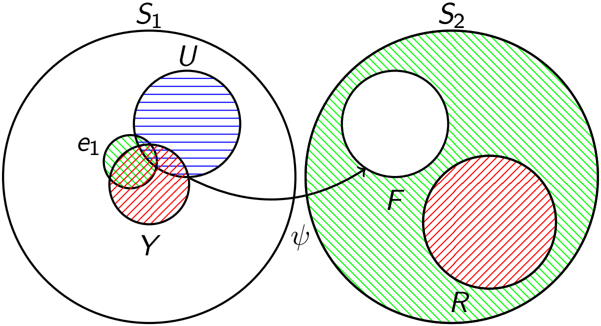
$$\int_{\tau^{\psi}(A)} f_1(\pi_{e_1}) f_2(\pi_{\psi(e_2)}) \cdots f_r(\pi_{\psi(e_r)})\, d\mu$$



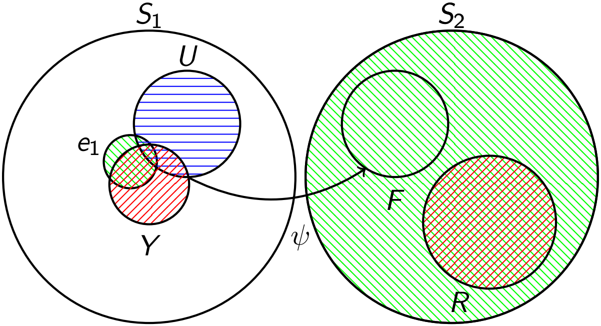
$$\int_{\tau^{\psi}(A)} \mathbb{E}\left[f_1(\pi_{e_1}) \middle| T^{S_2}\right] f_2(\pi_{\psi(e_2)}) \cdots f_r(\pi_{\psi(e_r)}) d\mu$$



$$\int_{\tau^{\psi}(A)} \mathbb{E}\left[f_1(\pi_{e_1}) \middle| T^{S_2\setminus(R\cup F)}\right] f_2(\pi_{\psi(e_2)}) \cdots f_r(\pi_{\psi(e_r)}) d\mu$$



$$\int_{A} \mathbb{E}\left[f_1(\pi_{e_1}) \middle| T^{S_2 \setminus (R \cup F)} \right] f_2(\pi_{e_2}) \cdots f_r(\pi_{e_r}) d\mu$$



$$\int_{A} \mathbb{E}\left[f_1(\pi_{e_1}) \middle| T^{S_2} \right] f_2(\pi_{e_2}) \cdots f_r(\pi_{e_r}) d\mu$$

Proof of Step 2 (cont'd).

In other words, we have:

$$
\begin{aligned}
&\int_A f_1(\pi_{e_1}) f_2(\pi_{e_2}) \cdots f_r(\pi_{e_r}) d\mu \\
&\quad = \int_{\tau^\psi(A)} f_1(\pi_{e_1}) f_2(\pi_{\psi(e_2)}) \cdots f_r(\pi_{\psi(e_r)}) d\mu \\
&\quad = \int_{\tau^\psi(A)} \mathbb{E}\left[f_1(\pi_{e_1}) \middle| T^{S_2} \right] f_2(\pi_{\psi(e_2)}) \cdots f_r(\pi_{\psi(e_r)}) d\mu \\
&\quad = \int_{\tau^\psi(A)} \mathbb{E}\left[f_1(\pi_{e_1}) \middle| T^{S_2 \setminus (R \cup F)} \right] f_2(\pi_{\psi(e_2)}) \cdots f_r(\pi_{\psi(e_r)}) d\mu \\
&\quad = \int_A \mathbb{E}\left[f_1(\pi_{e_1}) \middle| T^{S_2 \setminus (R \cup F)} \right] f_2(\pi_{e_2}) \cdots f_r(\pi_{e_r}) d\mu \\
&\quad = \int_A \mathbb{E}\left[f_1(\pi_{e_1}) \middle| T^{S_2} \right] f_2(\pi_{e_2}) \cdots f_r(\pi_{e_r}) d\mu.
\end{aligned}
$$

This concludes the proof of this step. □

Proof of Step 3.

Choose any infinite set $V \subseteq S_2$ such that $S_2 \setminus V$ is also infinite. Now let $\psi : S \to \{s,t\} \cup S_2$ be a bijection with the following properties: $\psi(s) = s, \psi(t) = t$, $\psi(S_2) = V$, and $\psi(S_1 \setminus \{s,t\}) = S_2 \setminus V$. By exchangeability, we have the following for any bounded measurable $f : K \to \mathbb{R}$:

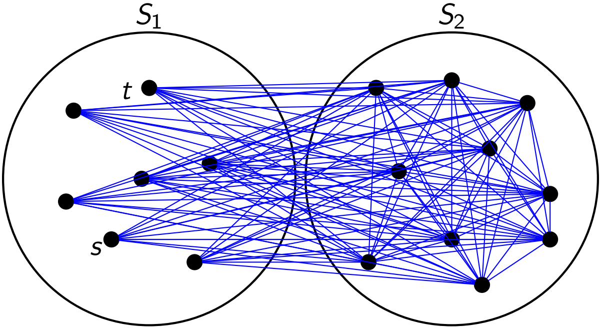
$$\mathbb{E}\left[f(\pi_{\{s,t\}}) \middle| T^{S_2}\right] \overset{d}{=} \mathbb{E}\left[f(\pi_{\{s,t\}}) \middle| \tau^{\psi}(T^{S_2})\right]. \tag{11}$$

In particular, this gives

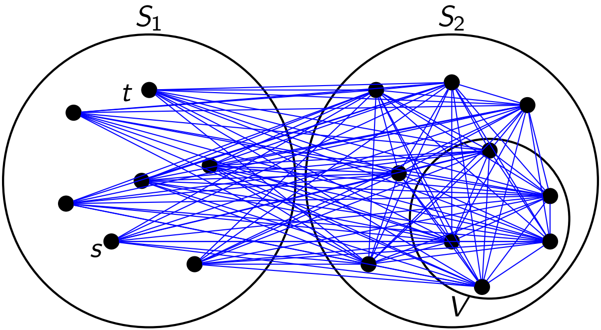
$$\mathbb{E}\left|\mathbb{E}\left[f(\pi_{\{s,t\}}) \middle| T^{S_2}\right]\right|^2 = \mathbb{E}\left|\mathbb{E}\left[f(\pi_{\{s,t\}}) \middle| \tau^{\psi}(T^{S_2})\right]\right|^2. \tag{12}$$

Moreover, the construction of ψ implies that we get the inclusions $\tau^{\psi}(T^{S_2}) \subseteq \sigma(\pi_s^{S_2}, \pi_t^{S_2}) \subseteq T^{S_2}$. The second of these inclusions is obvious, and the first is easily proven. For some intuition, consider the following graphic:

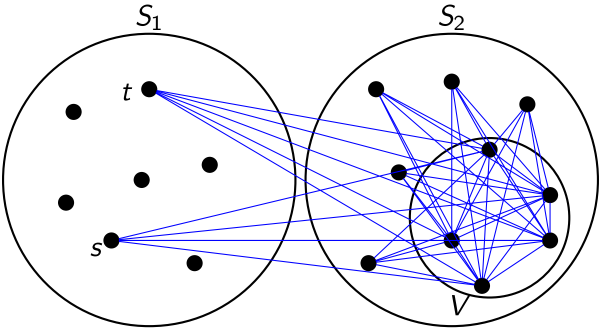
T^{S_2}



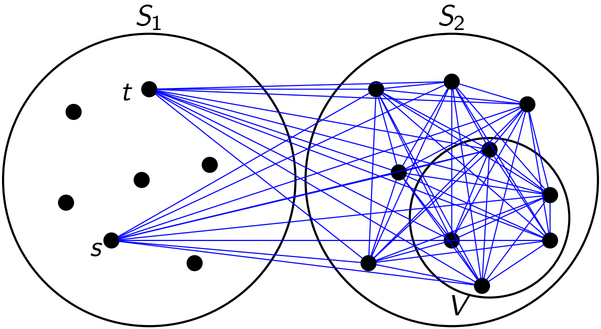
T^{S_2}



$\tau^{\psi}(T^{S_2})$



$\sigma(\pi_s^{S_2}, \pi_t^{S_2})$



Proof of Step 3 (cont'd).

More precisely, it suffices to show $\tau^{\psi}(T^{S_2}) \subseteq \sigma(\pi_s^{S_2}, \pi_t^{S_2})$ by checking this on a generating set. Indeed, if $A \in \sigma(\pi_e; e \in \binom{\{y_j\}\cup R_j}{2}, 1 \leq j \leq m)$ for $y_j \in S_1$ and $R_j \subseteq S_2$ finite for all $1 \leq j \leq m$, then we have $\psi(\{y_1, \ldots y_m\} \cup \bigcup_{j=1}^m R_j) \subseteq \{s, t\} \cup S_2$, so $\tau^{\psi}(A) \in \sigma(\pi_s^{S_2}, \pi_t^{S_2})$ follows.

Now by (12) and two applications of the contraction property of conditional expectation, we get

$$\begin{aligned}
\mathbb{E}\left|\mathbb{E}\left[f(\pi_{\{s,t\}})\middle| T^{S_2}\right]\right|^2 &\geq \mathbb{E}\left|\mathbb{E}\left[f(\pi_{\{s,t\}})\middle|\sigma(\pi_s^{S_2}, \pi_t^{S_2})\right]\right|^2 \\
&\geq \mathbb{E}\left|\mathbb{E}\left[f(\pi_{\{s,t\}})\middle|\tau^{\psi}(T^{S_2})\right]\right|^2 \\
&= \mathbb{E}\left|\mathbb{E}\left[f(\pi_{\{s,t\}})\middle| T^{S_2}\right]\right|^2.
\end{aligned}$$

By energy-squeezing, this implies that $\mathbb{E}[f(\pi_{\{s,t\}})| T^{S_2}]$ is $\sigma(\pi_s^{S_2}, \pi_t^{S_2})$-measurable, as claimed. □

Proof of Step 4.

Let $\psi : S_1 \to S_1$ be any finite permutation. Now let $s_1, \dots s_r \in S_1$ be arbitrary, and take some sets $B_1, \dots B_r \in \mathcal{B}(K^{\binom{\{*\}\cup S_2}{2}})$. By exchangeability of μ, we get

$$\begin{aligned}
&\mathbb{P}\left(\pi_{s_1}^{S_2} \in B_1, \dots \pi_{s_r}^{S_2} \in B_r\right) \\
&= \mathbb{P}\left(\{\pi_e\}_{e\in\binom{\{s_1\}\cup S_2}{2}} \in B_1, \dots \{\pi_e\}_{e\in\binom{\{s_r\}\cup S_2}{2}} \in B_r\right) \\
&= \mathbb{P}\left(\{\pi_e\}_{e\in\binom{\{\psi(s_1)\}\cup S_2}{2}} \in B_1, \dots \{\pi_e\}_{e\in\binom{\{\psi(s_r)\}\cup S_2}{2}} \in B_r\right) \\
&= \mathbb{P}\left(\pi_{\psi(s_1)}^{S_2} \in B_1, \dots \pi_{\psi(s_r)}^{S_2} \in B_r\right)
\end{aligned}$$

because ψ fixes S_2, and this proves the claim. □

Finishing the Proof.

Set $Z_1 = K^{S_2} \times K^{\binom{S_2}{2}}$ so that $\{\pi_s^{S_2}\}_{s\in S_1}$ is a Z_1-coloring of S_1, and let μ_1 be the law of $\{\pi_s^{S_2}\}_{s\in S_1}$ on $Z_1^{S_1}$. By Step 2, there exists a family of kernels $\{P_e\}_{e\in\binom{S_1}{2}}$ from $Z_1^{S_1}$ to K such that the conditional distribution of $\{\pi_e\}_{e\in\binom{S_1}{2}}$ given T^{S_2} is distributed according to $\bigotimes_{e\in\binom{S_1}{2}} P_e(\{\pi_s^{S_2}\}_{s\in S_1},\cdot)$. But by Step 3, there exists a kernel $\tilde{P}_2 : Z_1^2 \to K$ such that all of the marginal kernels can be written as $P_{\{s,t\}}(\{\pi_s^{S_2}\}_{s\in S_1},\cdot) = \tilde{P}_2(\pi_s^{S_2},\pi_t^{S_2},\cdot)$. That is, the conditional distribution of $\{\pi_e\}_{e\in\binom{S_1}{2}}$ given T^{S_2} is distributed according to $\bigotimes_{\{s,t\}\in\binom{S_1}{2}} \tilde{P}_2(\pi_s^{S_2},\pi_t^{S_2},\cdot)$.

By Step 4, the law μ_1 is exchangeable on $Z_1^{S_1}$. Hence, by de Finetti's theorem, there exists (Z_0,μ_0) and a kernel $P_1 : Z_0 \to Z_1$ such that $\{\pi_s^{S_2}\}_{s\in S_1}$ are distributed according to $P_1^{\otimes S_1}(z_\emptyset,\cdot)$. Now set $P_2(z_\emptyset, z_s, z_t,\cdot) = \tilde{P}_2(z_s,z_t,\cdot)$.

Matching definitions, we see that this means that the ingredients $(Z_0,\mu_0),(Z_1,P_1),(K,P_2)$ yield μ upon following the standard recipe. □

Exchangeable Hyperedge-Colorings, $k \geq 2$

Set-Up

- In order to prove the general result, we will prove a slightly more general statement, in which the space of possible colorings is allowed to change at every level, and then we'll deduce the desired structure theorem as a special case. Moving to the expanded statement can really just be seen as strenghtening the inductive hypothesis in our main proof by induction.
- By a *palette* we mean a finite sequence of standard Borel spaces $\mathbf{K} = (K_0, \ldots K_k)$ where $k \in \mathbb{N}$ is called the *rank*.
- A $\mathbf{K}$-coloring of an infinite set S is a sequence of maps $(H_0, \ldots H_k)$ such that H_i is a K_i-coloring of $\binom{S}{i}$ for each $i = 0, \ldots k$. The space of all $\mathbf{K}$-colorings of S is denoted $\mathbf{K}^{(S)} = \prod_{i=0}^{k} K_i^{\binom{S}{i}}$.
- A probability measure on $\mathbf{K}^{(S)}$ is called a *random* $\mathbf{K}$*-coloring of* S. If μ is invariant under finite permutations of S, then we call it *exchangeable*. As usual, we can also think of μ via its canonical coordinate process.

The Standard Recipe

- A collection of *ingredients* is a sequence of standard Borel spaces $Z_0, \ldots Z_{k-1}, Z_k = K$, a probability measure μ_0 on Z_0, a collection of probability kernels $P_1, \ldots P_k$ such that P_i is a kernel from $Z_0 \times Z_1^i \times Z_2^{\binom{i}{2}} \times \cdots \times Z_{i-1}^{\binom{i}{i-1}}$ to Z_i with each P_i invariant under finite permutations of S, and a sequence of measurable maps $\kappa_0, \ldots \kappa_k$ such that $\kappa_i : Z_i \to K_i$.
- The *standard recipe* is the following recursive algorithm for sampling a **K**-coloring of S:
  - Base Case: Sample a random $z_\emptyset \in Z_0$ according to μ_0.
  - Inductively for $2 \leq i \leq k$: For each hyperedge $u \in \binom{S}{i}$, sample $z_u \in Z_i$ according to the law

$$P_i\left(z_\emptyset, z_u, \{z_a\}_{a \in \binom{u}{2}}, \cdots \{z_a\}_{a \in \binom{u}{i-1}}, \ \cdot \ \right). \tag{13}$$

  - Define the colorings $H_i = \{\kappa_i(z_u)\}_{u \in \binom{S}{i}}$ for all $i = 0, \ldots k-1$, and $H_k = \{z_u\}_{u \in \binom{S}{k}}$. (That is, we set $\kappa_k = \mathrm{Id}_K$ by convention.)

Structure Theorem

If we start with any collection of ingredients, then it is clear that the output of the standard recipe is an exchangeable random $\mathbf{K}$-coloring of S. As before, our goal is to show that this actually exhausts all possible examples:

Theorem (3.10 of [Aus08])

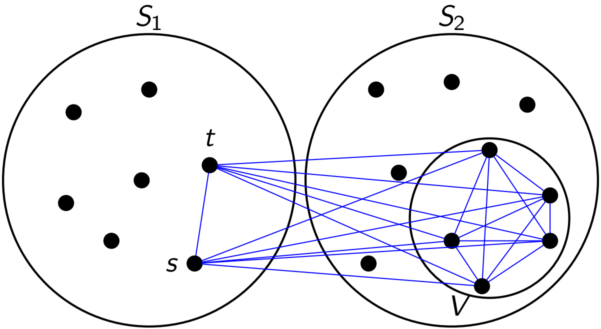
If $\mathbf{K} = (K_0, \dots K_k)$ is a palette and μ is any exchangeable random $\mathbf{K}$-coloring of an infinite set S, then there exists a collection of ingredients $(Z_0, \mu_0, \kappa_0), (Z_1, P_1, \kappa_1), \cdots (Z_{k-1}, P_{k-1}, \kappa_{k-1}), (K, P_k)$ such that the output of the standard recipe with these ingredients is a random element of $\mathbf{K}^{(S)}$ with law μ.

This will immediately imply the main structure theorem of interest by considering the palette $\mathbf{K} = (\{*\}, \dots \{*\}, K)$ for $\{*\}$ any one-point space.

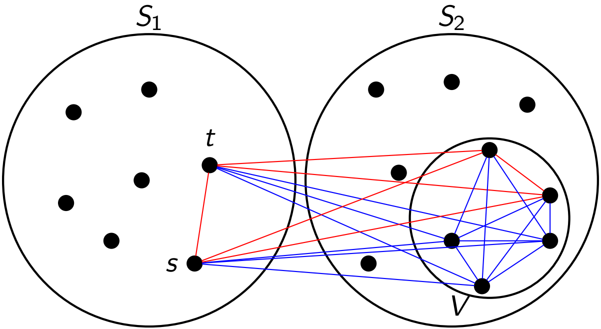
Proof Outline

- **Set-Up:** Partition S into two infinite sets $S_1 \cup S_2$. For any $a \in \binom{S_1}{\leq k-1}$ and $V \subseteq S_2$, set $\overline{\pi}_a^V = \{\pi_{a \cup v} : v \in \binom{V}{k-|a|}\}$, which is a random element of $K_k^{\binom{S_2}{k-|a|}}$. For convenience, also write $\tilde{\pi}_a = (\pi_a, \overline{\pi}_a^{S_2})$. Now set $\mathcal{T}^V = \sigma(\tilde{\pi}_a : a \in \binom{S_1}{\leq k-1})$ for $V \subseteq S_2$. (See next slides for a visualization.)
- **Step 1.** Show that, for any $u \in \binom{S_1}{k}$ and any infinite $V \subseteq S_2$, the conditional distribution of π_u given $\mathcal{T}^{S_2}$ is $\mathcal{T}^V$-measurable.
- **Step 2.** Show that the random variables $\{\pi_u : u \in \binom{S_1}{k}\}$ are conditionally independent given $\mathcal{T}^{S_2}$
- **Step 3.** Show that, for any $u \in \binom{S_1}{k}$, the conditional distribution of π_u given $\mathcal{T}^{S_2}$ is $\sigma(\tilde{\pi}_a ; a \in \binom{u}{\leq k-1})$-measurable.
- **Step 4.** Show that $\{\tilde{\pi}_a\}_{a \in \binom{S_1}{\leq k-1}}$ is an exchangeable $\tilde{\mathbf{K}}$-coloring of S, where $\tilde{\mathbf{K}}$ is the palette $(\tilde{K}_0, \ldots \tilde{K}_{k-1})$ for $\tilde{K}_i = K_i \times K_k^{\binom{S_2}{k-i}}$.

$k = 4, \ \overline{\pi}^{V}_{\{s,t\}}$



$k = 4, \ \overline{\pi}^{V}_{\{s,t\}}$



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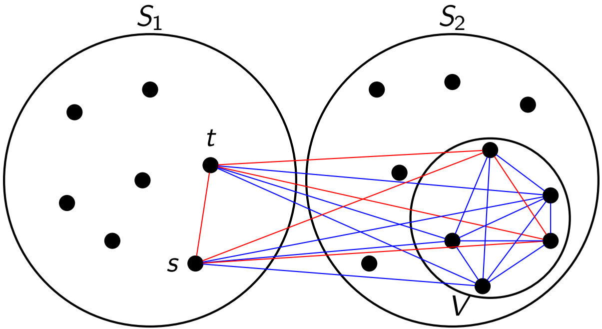
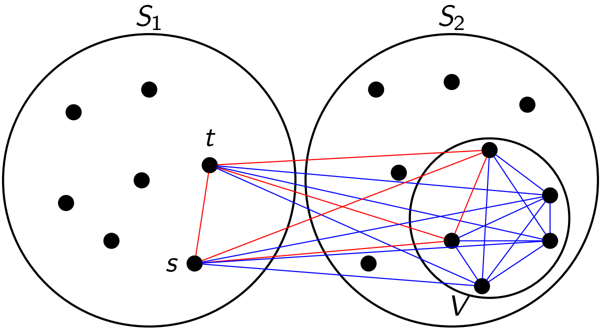
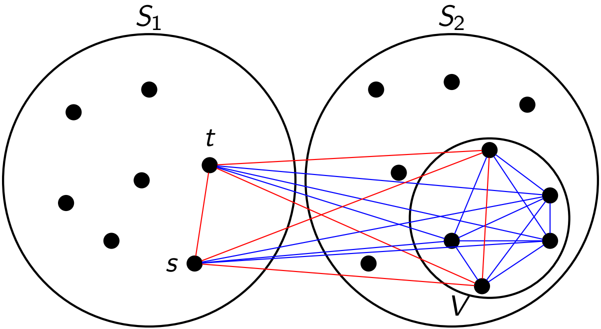


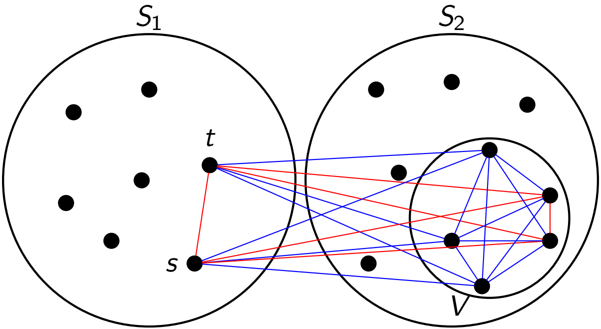
$k = 4, \ \overline{\pi}^{V}_{\{s,t\}}$



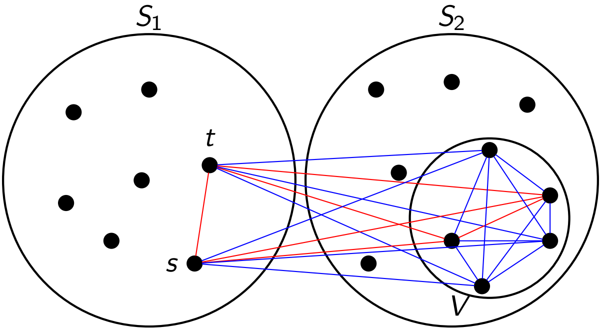
$k = 4,\ \overline{\pi}^{V}_{\{s,t\}}$



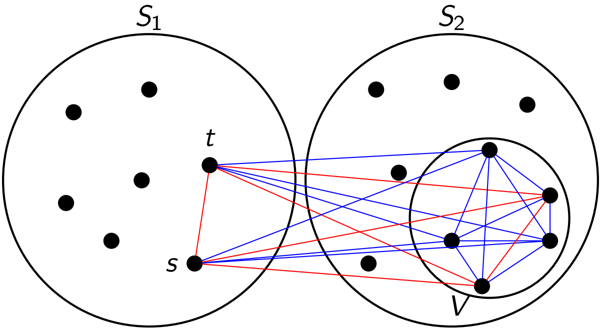
$k = 4, \ \overline{\pi}^{V}_{\{s,t\}}$



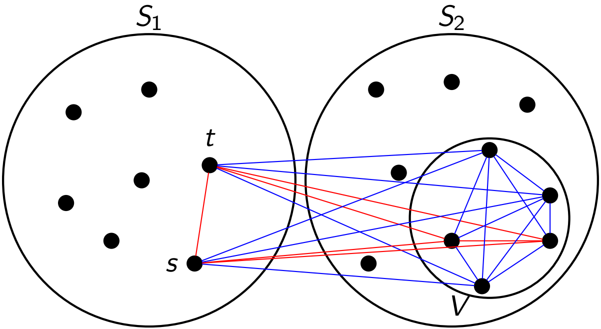
$k = 4,\ \overline{\pi}^{V}_{\{s,t\}}$



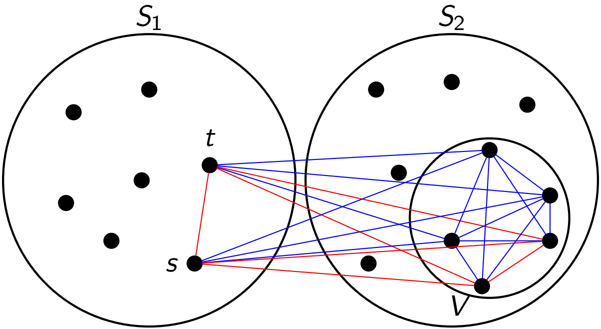
$k = 4, \ \overline{\pi}^{V}_{\{s,t\}}$



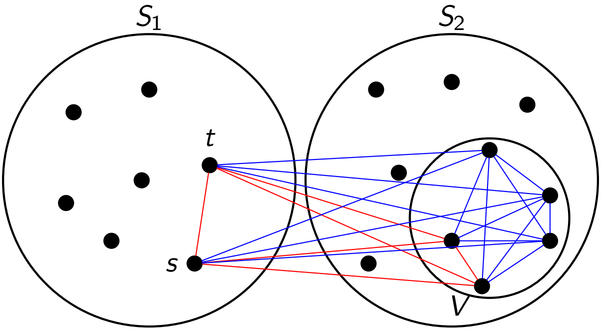
$k = 4, \; \overline{\pi}^{V}_{\{s,t\}}$



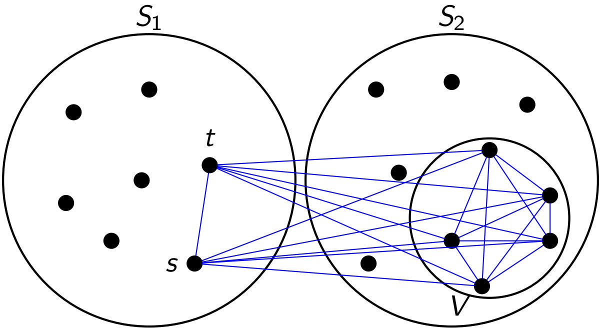
$k = 4,\ \overline{\pi}^{V}_{\{s,t\}}$



$k = 4, \ \overline{\pi}^{V}_{\{s,t\}}$



$k = 4, \ \overline{\pi}^{V}_{\{s,t\}}$



Proof of Step 1.

The proof is the same as for $k \leq 2$. For $u \in \binom{S_1}{k}$, we have:

$$\mathbb{E}\left|\mathbb{E}\left[f(\pi_u)\middle|T^V\right]\right|^2$$
$$= \sup_{\substack{F \subseteq V \\ F \text{finite}}} \mathbb{E}\left|\mathbb{E}\left[f(\pi_e)\middle|\sigma\left(\pi_a; a \in \binom{u \cup F}{k} \setminus u\right)\right]\right|^2.$$

By exchangeability, the right side depends only on $|F|$, because we can swap F with any set in V of the same size. As before, V and S_2 both contain arbitrarily large subsets, so

$$\mathbb{E}\left|\mathbb{E}\left[f(\pi_u)\middle|T^V\right]\right|^2 = \mathbb{E}\left|\mathbb{E}\left[f(\pi_u)\middle|T^{S_2}\right]\right|^2. \tag{14}$$

Hence, energy-squeezing implies that $\mathbb{E}[f(\pi_u)|T^{S_2}]$ is T^V-measurable. □

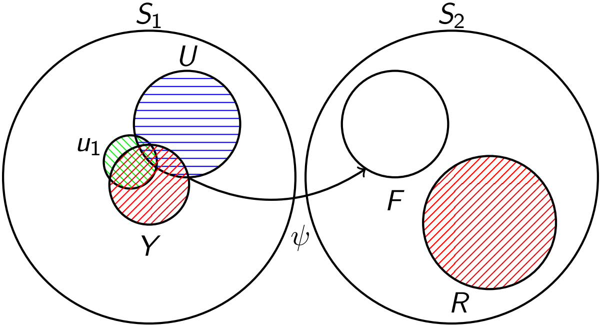
Proof of Step 2.

The proof is similar to the $k = 2$ case; take any bounded measurable functions $f_1, \dots f_r : K \to \mathbb{R}$ and any fixed $u_1, \dots u_r \in \binom{S_1}{k}$. By induction on r and approximation by cylinder sets, it suffices to show that we have

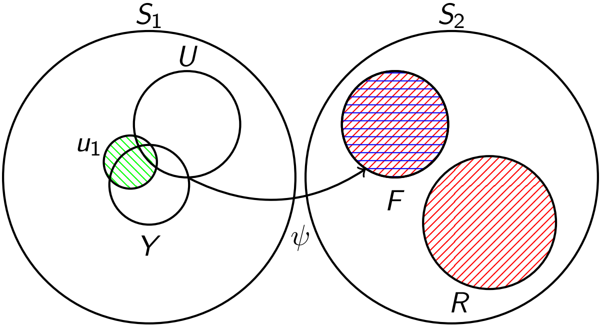
$$\int_A f_1(\pi_{u_1}) \cdots f_r(\pi_{u_r}) d\mu = \int_A \mathbb{E}\left[f_1(\pi_{u_1}) \middle| T^{S_2} \right] f_2(\pi_{u_2}) \cdots f_r(\pi_{u_r}) d\mu, \quad (15)$$

where A is $\sigma(\pi_u; u \in \binom{y_j \cup R_j}{\leq k}, 1 \leq j \leq m)$-measurable, for $y_j \in \binom{S_1}{k-1}$ and $R_j \subseteq S_2$ is finite for all $1 \leq j \leq m$. Then let $Y = \bigcup_{i=1}^m y_i$, $R = \bigcup_{i=1}^m R_i$, and $U = \bigcup_{i=2}^m u_i$. Let ψ be the (finite) permutation that swaps $(U \cup Y) \setminus u_1$ with some finite set of vertices $F \subseteq S_2 \setminus R$ (and hence which leaves the vertices in u_1 fixed).

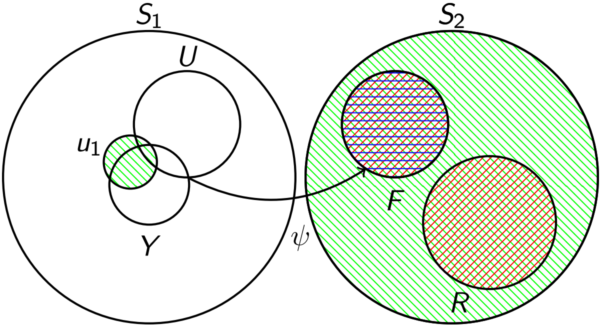
Now we apply exchangeability and Step 1 many times to get the usual string of equalities:



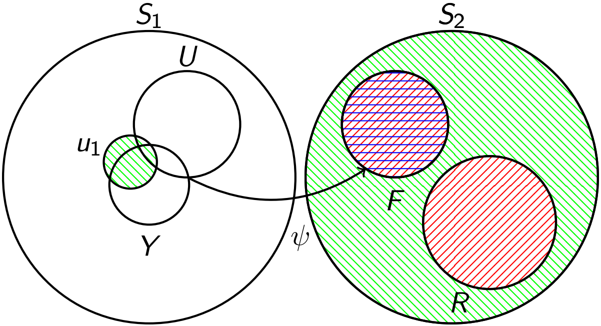
$$\int_A f_1(\pi_{u_1}) f_2(\pi_{u_2}) \cdots f_r(\pi_{u_r}) d\mu$$



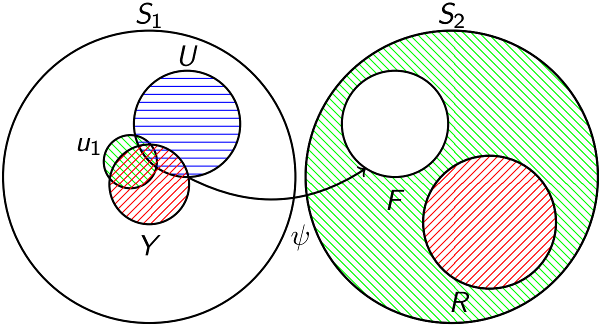
$$\int_{\tau^{\psi}(A)} f_1(\pi_{u_1}) f_2(\pi_{\psi(u_2)}) \cdots f_r(\pi_{u_r})\, d\mu$$



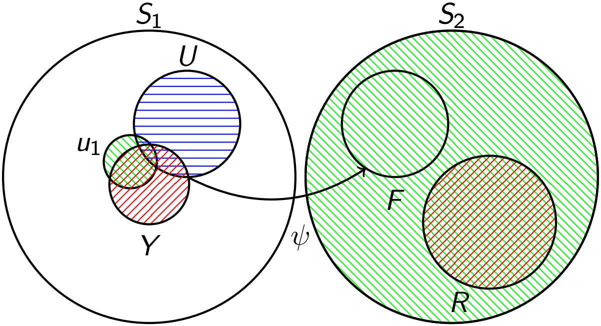
$$\int_{\tau^{\psi}(A)} \mathbb{E}\left[f_1(\pi_{u_1}) \middle| T^{S_2}\right] f_2(\pi_{\psi(u_2)}) \cdots f_r(\pi_{\psi(u_r)}) d\mu$$



$$\int_{\tau^{\psi}(A)} \mathbb{E}\left[f_1(\pi_{u_1})\middle| T^{S_2\setminus(R\cup F)}\right] f_2(\pi_{\psi(u_2)})\cdots f_r(\pi_{\psi(u_r)})d\mu$$



$$\int_A \mathbb{E}\left[f_1(\pi_{u_1})\middle| T^{S_2\setminus(R\cup F)}\right] f_2(\pi_{u_2})\cdots f_r(\pi_{u_r})d\mu$$



$$\int_A \mathbb{E}\left[f_1(\pi_{u_1})\middle| T^{S_2}\right] f_2(\pi_{u_2})\cdots f_r(\pi_{u_r})\,d\mu$$

Proof of Step 2 (cont'd).

As usual, this shows

$$
\begin{aligned}
&\int_A f_1(\pi_{u_1}) f_2(\pi_{u_2}) \cdots f_r(\pi_{u_r}) d\mu \\
&\quad = \int_{\tau^\psi(A)} f_1(\pi_{u_1}) f_2(\pi_{\psi(u_2)}) \cdots f_r(\pi_{\psi(u_r)}) d\mu \\
&\quad = \int_{\tau^\psi(A)} \mathbb{E}\left[f_1(\pi_{u_1}) \middle| T^{S_2} \right] f_2(\pi_{\psi(u_2)}) \cdots f_r(\pi_{\psi(u_r)}) d\mu \\
&\quad = \int_{\tau^\psi(A)} \mathbb{E}\left[f_1(\pi_{u_1}) \middle| T^{S_2 \setminus (R \cup F)} \right] f_2(\pi_{\psi(u_2)}) \cdots f_r(\pi_{\psi(u_r)}) d\mu \\
&\quad = \int_A \mathbb{E}\left[f_1(\pi_{u_1}) \middle| T^{S_2 \setminus (R \cup F)} \right] f_2(\pi_{u_2}) \cdots f_r(\pi_{u_r}) d\mu \\
&\quad = \int_A \mathbb{E}\left[f_1(\pi_{u_1}) \middle| T^{S_2} \right] f_2(\pi_{u_2}) \cdots f_r(\pi_{u_r}) d\mu,
\end{aligned}
$$

so the result follows. □

Proof of Step 3.

Again, we follow the idea of the $k = 2$ case. Choose any infinite set $V \subseteq S_2$ such that $S_2 \setminus V$ is also infinite, and let $\psi : S \to u \cup S_2$ be a bijection satisfying: $\psi(u) = u$, $\psi(S_2) = V$, and $\psi(S_1 \setminus u) = S_2 \setminus V$. By exchangeability, we have the following for any bounded measurable $f : K \to \mathbb{R}$:

$$\mathbb{E}\left[f(\pi_u)\middle| T^{S_2}\right] \overset{d}{=} \mathbb{E}\left[f(\pi_u)\middle|\tau^{\psi}(T^{S_2})\right], \tag{16}$$

which implies

$$\mathbb{E}\left|\mathbb{E}\left[f(\pi_u)\middle| T^{S_2}\right]\right|^2 = \mathbb{E}\left|\mathbb{E}\left[f(\pi_u)\middle|\tau^{\psi}(T^{S_2})\right]\right|^2. \tag{17}$$

Of course, we still have $\sigma(\tilde{\pi}_a; a \in \binom{u}{\leq k-1}) \subseteq T^{S_2}$. To show that we have $\tau^{\psi}(T^{S_2}) \subseteq \sigma(\tilde{\pi}_a; a \in \binom{u}{\leq k-1})$, we use a similar method to the case of $k = 2$. (The visualization of this step breaks down a bit, since $\sigma(\tilde{\pi}_a; a \in \binom{u}{\leq k-1})$ contains many hyperedges of different sizes.)

Proof of Step 3 (cont'd).

We will prove $\tau^\psi(T^{S_2}) \subseteq \sigma(\tilde{\pi}_a; a \in \binom{u}{\leq k-1})$ by checking it on a generating set. So, suppose that $A \in \sigma(\pi_a; a \in \binom{y_j \cup R_j}{\leq k}, 1 \leq j \leq m)$ for $y_j \in \binom{S_1}{k-1}$ and $R_j \subseteq S_2$ finite for all $1 \leq j \leq m$. Adopting similar language as in the proof of Step 2, define $Y = \bigcup_{j=1}^m y_j$ and $R = \bigcup_{j=1}^m R_j$. Now observe that we have $\psi(Y \cup R) \subseteq u \cup S_2$, so $\tau^\psi(A) \in \sigma(\tilde{\pi}_a; a \in \binom{u}{\leq k-1})$ follows. As before, this implies

$$\begin{aligned}
\mathbb{E}\left|\mathbb{E}\left[f(\pi_u)\middle| T^{S_2}\right]\right|^2 &\geq \mathbb{E}\left|\mathbb{E}\left[f(\pi_u)\middle|\sigma\left(\tilde{\pi}_a; a \in \binom{u}{\leq k-1}\right)\right]\right|^2 \\
&\geq \mathbb{E}\left|\mathbb{E}\left[f(\pi_u)\middle|\tau^\psi(T^{S_2})\right]\right|^2 \\
&= \mathbb{E}\left|\mathbb{E}\left[f(\pi_u)\middle| T^{S_2}\right]\right|^2.
\end{aligned}$$

so energy-squeezing shows that $\mathbb{E}[f(\pi_u)|T^{S_2}]$ is $\sigma(\tilde{\pi}_a; a \in \binom{u}{\leq k-1})$-measurable, as claimed. □

Proof of Step 4.

Let $\psi : S_1 \to S_1$ be any finite permutation, and let $a_1, \dots a_r \in \binom{S_1}{\leq k-1}$ be arbitrary. Recall that $\tilde{\pi}_{a_i} = (\pi_{s_1}, \overline{\pi}^{S_2}_{a_i}) \in K_{|a_i|} \times K_k^{\binom{S_2}{k-|a_i|}}$, so take arbitrary sets $B_1, \dots B_r$ and $\overline{B}_1, \dots \overline{B}_r$ with $B_i \in \mathcal{B}(K_{|a_i|})$ and $\overline{B}_i = \mathcal{B}(K_k^{\binom{S_2}{k-|a_i|}})$ for $1 \leq i \leq r$, and set $\tilde{B}_i = B_i \times \overline{B}_i$. By exchangeability of μ, we get

$$\begin{aligned}
&\mathbb{P}\left(\tilde{\pi}_{a_1} \in \tilde{B}_1, \dots \tilde{\pi}_{a_r} \in \tilde{B}_r\right) \\
&= \mathbb{P}\left(\bigcap_{i=1}^{r}\left\{\pi_{a_i} \in B_i, \{\pi_{a_i \cup v}\}_{v \in \binom{a_i \cup S_2}{k-|a_i|}} \in \overline{B}_i\right\}\right) \\
&= \mathbb{P}\left(\bigcap_{i=1}^{r}\left\{\pi_{\psi(a_i)} \in B_i, \{\pi_{\psi(a_i) \cup v}\}_{v \in \binom{\psi(a_i) \cup S_2}{k-|\psi(a_i)|}} \in \overline{B}_i\right\}\right) \\
&= \mathbb{P}\left(\tilde{\pi}_{\psi(a_1)} \in \tilde{B}_1, \dots \tilde{\pi}_{\psi(a_r)} \in \tilde{B}_r\right)
\end{aligned}$$

because ψ fixes S_2, and this proves the claim. □

Finishing the Proof.

The proof is by induction on the rank k of the palette, so the base case $k = 1$ corresponds to de Finetti's theorem which we have already proven. For the inductive step, suppose that $\mathbf{K}$ is a palette of rank $k \geq 2$.

By Step 2, there exists a family of kernels $\{P_u\}_{u \in \binom{S_1}{k}}$ from $\tilde{\mathbf{K}}^{(S)}$ to K_k such that the conditional distribution of $\{\pi_u\}_{u \in \binom{S_1}{k}}$ given T^{S_2} is distributed according to $\bigotimes_{u \in \binom{S_1}{k}} P_u(\{\tilde{\pi}_a\}_{a \in \binom{S_1}{\leq k-1}}, \cdot\,)$. But by Step 3, there exists a single kernel $P_k : \tilde{\mathbf{K}}^{([k-1])} \to K_k$ such that all of the marginal kernels can be written as $P_u(\{\tilde{\pi}_a\}_{a \in \binom{S_1}{\leq k-1}}, \cdot) = P_k(\{\tilde{\pi}_a\}_{a \in \binom{u}{\leq k-1}}, \cdot)$. That is, the conditional distribution of $\{\pi_u\}_{u \in \binom{S_1}{k}}$ given T^{S_2} is distributed according to $\bigotimes_{u \in \binom{S_1}{k}} P_k(\{\tilde{\pi}_a\}_{a \in \binom{u}{\leq k-1}}, \cdot)$.

Finishing the Proof (cont'd).

Now let $\tilde{\mu}$ be the law of $\{(\pi_u, \overline{\pi}_u^{S_2})\}_{u \in \binom{S_1}{\leq k-1}}$ on $\tilde{\mathbf{K}}^{(S)}$. By Step 4, $\tilde{\mu}$ is exchangeable, so the inductive hypothesis guarantees a sequence of ingredients $(Z_0, \mu_0, \tilde{\kappa}_0), \ldots (Z_{k-2}, P_{k-2}, \tilde{\kappa}_{k-2}), (Z_{k-1}, P_{k-1})$ which give rise to $\tilde{\mu}$ when applying the standard recipe. Write $\overline{\kappa}_i : \tilde{K}_i = K_i \times K_k^{\binom{S_2}{k-i}} \to K_i$ for the projection onto the K_i coordinate, and set $\kappa_i = \overline{\kappa}_i \circ \tilde{\kappa}_i : Z_i \to K_i$. Unpacking definitions, this implies that $(Z_0, \mu_0, \kappa_0), \ldots (Z_{k-1}, P_{k-1}, \kappa_{k-1}), (Z_k, P_k)$ is a sequence of ingredients for which the output of the standard recipe is μ. This proves the structure theorem. □

Further Ideas

Related Results

There are many possible refinements and extensions of the structure theorem we just proved.

- There is a sense in which the determination of ingredients (Z_0, μ_0), $(Z_1, P_1), \ldots (Z_{k-1}, P_{k-1}), (K, P_k)$ depends continuously on μ
- Analogous statements can be proved for *directed uniform hypergraphs* and for *partite uniform hypergraphs*
- Similar results hold for *spreadable* hypergraph colorings (where the general definition of "spreadable" reduces to the usual case for $S = \mathbb{N}$.) It turns out that spreadability and exchangeability are not generally equivalent in the hypergraph setting.
- It can be proven by example that the added randomness at every level is necessary for the result to hold: There exist exchangeable colorings of a 3-uniform hypergraph which cannot be written through a vertex-sampling construction.

Equivalent Results

The main structure theorem is equivalent to some historically earlier results. For example, note that colored undirected graphs and random symmetric arrays are two ways of encoding the same information. Then the following result is equivalent to the structure theorem when $k = 2$:

Theorem (14.11 of [Ald85])

Let $= \{X_{i,j} : i, j \geq 1\}$ *be a symmetric array of* K*-valued random variables such that, for any finite permutation* σ *of indices, we have*

$$\{X_{i,j} : i, j \geq 1\} \stackrel{d}{=} \{X_{\sigma(i),j} : i, j \geq 1\} \stackrel{d}{=} \{X_{i,\sigma(j)} : i, j \geq 1\}. \tag{18}$$

Then there exists a measurable function $f : [0,1]^4 \to K$ *with* $f(\cdot, u, v, \cdot) = f(\cdot, v, u, \cdot)$ *such that, if* $\{\omega, U_i, V_j, \xi_{i,j} : i, j \geq 1\}$ *is a collection of iid uniform* $[0,1]$*-valued random variables, then the law of* $\{f(\omega, U_i, V_i, \xi_{i,j})\}_{i,j\geq 1}$ *equals the law of* $\mathbb{X}$*.*

Suitable adaptation of this idea leads to an equivalent statement for $k \geq 2$.

Applications and Interpretations

The structure theorem for exchangeable colorings of uniform hypergraphs is related to many ideas across many mathematical disciplines.

- Statistical testability of graph and hypergraph properties
- Extremal graph theory and graph limits
- Model theory
- Arithmetic combinatorics, in which the structure theorem is a way to connect two seemingly-disparate proofs of Szemeredi's theorem: one via graph theory and one via ergodic theory

See [Aus08] for more detail and for a long list of these ideas.

Thank you!

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