STAT GU4207 / GR5207 FINAL EXAM

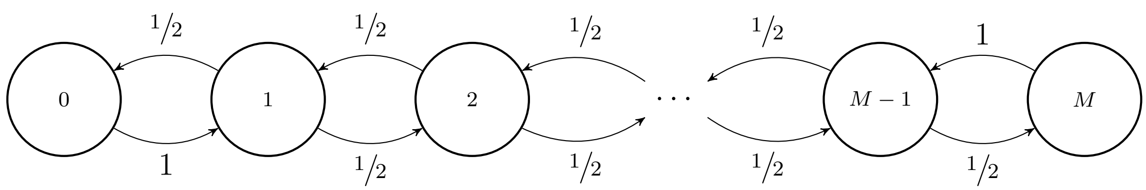
SPRING 2025

1. (10 points) For $0 < \alpha < 1$, let $\{X_n\}_{n \geq 0}$ be a discrete-time Markov chain on the state space $\{0, 1, 2, \ldots\}$ with transition matrix given by

$$P = \begin{pmatrix} 1-\alpha & \alpha & 0 & 0 & \cdots \\ 1-\alpha & 0 & \alpha & 0 & \cdots \\ 1-\alpha & 0 & 0 & \alpha & \cdots \\ 1-\alpha & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Show that $\{X_n\}_{n \geq 0}$ is irreducible.

2. (10 points) For $M \geq 0$, let $\{X_n\}_{n \geq 0}$ be the Markov chain on the state space $\{0, 1, \ldots, M\}$ represented by the following graph.



Compute $\mathbb{E}[T_M \,|\, X_0 = M]$, where $T_M := \min\{n \geq 1 : X_n = M\}$.

3. (10 points) Let $\{N_t\}_{t \geq 0}$ denote a Poisson process of rate $\lambda > 0$, and let $S_1, S_2, \ldots$ denote its arrival times. Compute

$$\mathbb{E}\left[(N_t - \lambda t) \sum_{k=1}^{N_t} S_k\right]$$

for $t \geq 0$.

4. (10 points) Let $\{S_1, S_2, \ldots\}$ be the points of a spatial Poisson point process in $\mathbb{R}^2$ with rate $\lambda > 0$, and let $\{M_t\}_{t \geq 0}$ be the counting process such that M_t is the number of points of $\{S_1, S_2, \ldots\}$ which are contained in the region

$$\{(s, x) : 0 \leq s \leq t \text{ and } 0 \leq x \leq t\}$$

for all $t \geq 0$. Show that $\{M_t\}_{t \geq 0}$ is a inhomogeneous Poisson process, and find its intensity function.

5. (10 points) Let $\lambda_1, \lambda_2, \ldots > 0$ satisfy $\lambda := \sum_{i=1}^{\infty} \lambda_i < \infty$, and let $\{X_t\}_{t\geq 0}$ denote a continuous-time Markov chain on the state space $\{0, 1, 2, \ldots\}$ with generator matrix

$$Q = \begin{pmatrix} -\lambda & \lambda_1 & \lambda_2 & \lambda_3 & \cdots \\ 1 & -1 & 0 & 0 & \cdots \\ 1/2 & 0 & -1/2 & 0 & \cdots \\ 1/3 & 0 & 0 & -1/3 & \cdots \\ 1/4 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Determine the $\lambda_1, \lambda_2, \ldots > 0$ for which this process is ergodic, and in this case finding the limiting distribution of $\{X_t\}_{t\geq 0}$.

6. (10 points) Suppose $\{X_n\}_{n\geq 0}$ is a discrete-time centered Gaussian process with covariance function $\Sigma(m, n) = \rho^{|m-n|}$ for $m, n \geq 0$, for some constant $0 < \rho < 1$. Find the conditional distribution of $\{X_n\}_{n\geq 0}$ given $\{X_0 = 0\}$.

7. (10 points) Let $\{X_n\}_{n\geq 0}$ be a discrete-time centered Gaussian process with covariance function $\Sigma : \{0, 1, 2, \ldots\} \times \{0, 1, 2, \ldots\} \to \mathbb{R}$, let $\{N_t\}_{t\geq 0}$ be a Poisson process of rate $\lambda > 0$, and let $\{X_n\}_{n\geq 0}$ and $\{N_t\}_{t\geq 0}$ be independent. Then define the stochastic process $\{\tilde{X}_t\}_{t\geq 0}$ by $\tilde{X}_t = X_{N_t}$ for $t \geq 0$. Show that $\{\tilde{X}_t\}_{t\geq 0}$ is a continuous-time Gaussian process if and only if there exists some $\sigma^2 > 0$ such that we have $\Sigma(m, n) = \sigma^2$ for all $m, n \geq 0$.

8. (10 points) Let $\{B_t\}_{t\geq 0}$ be a Brownian motion, and fix $T > 0$. Show that the process $\{\tilde{B}_t\}_{t\geq 0}$ defined via

$$\tilde{B}_t = B_{t+T} - B_T$$

is also a Brownian motion.