STAT GU4207 / GR5207 HOMEWORK #10

SPRING 2025

Assigned: Wed, Apr 2
Due: Fri, Apr 11

1. Find Gaussian random variables X and Y such that (X, Y) is not a multivariate Gaussian random vector.

2. Suppose that X, Y are independent centered Gaussian random variables with the same variance. Show that
$$\left(\frac{X+Y}{\sqrt{2}}, \frac{X-Y}{\sqrt{2}}\right)$$
is a multivariate Gaussian random vector with the same distribution as (X, Y).

3. Fix $X_0 = X_1 = 1$, and define the *Fibonacci process* $\{X_n\}_{n\geq 0}$ via
$$X_n = X_{n-1} + X_{n-2} + \varepsilon_n$$
for $n \geq 2$, where $\varepsilon_2, \varepsilon_3, \ldots$ are independent identically-distributed samples from $\mathcal{N}(0, \sigma^2)$. Show that $\{X_n\}_{n\geq 0}$ is a discrete-time Gaussian process. (Also, what do you think happens to X_n as $n \to \infty$?)

4. Fix reals θ_1, θ_2 and $\sigma^2 > 0$, and suppose that $\varepsilon_1, \varepsilon_2, \ldots$ are independent identically-distributed samples from $\mathcal{N}(0, \sigma^2)$. Define the second-order Gaussian autoregressive process $\{X_n\}_{n\geq 0}$ via
$$X_n = \theta_1 X_{n-1} + \theta_2 X_{n-2} + \varepsilon_n$$
for $n \geq 0$, where X_{-2}, X_{-1} are fixed and known. Show that $\{X_n\}_{n\geq 0}$ is a Markov chain if and only if $\theta_2 = 0$.

5. Suppose that $(X_1, \ldots, X_m) \sim \mathcal{N}(\mu, \Sigma)$ is a multivariate Gaussian random vector, and that I is a random variable which is uniformly distributed on $\{1, \ldots, m\}$ and which is independent of $(X_1, \ldots, X_m)$. Determine the μ and Σ for which the random vector
$$(X_1, \ldots, X_{I-1}, X_{I+1}, \ldots, X_m)$$
is a multivariate Gaussian random vector.