STAT GU4207 / GR5207 HOMEWORK #11

SPRING 2025

Assigned: Wed, Apr 9
Due: Fri, Apr 18

1. Let $\{X_t\}_{t\geq 0}$ be Gaussian process with mean function μ and covariance function Σ, and let $\theta : [0,\infty) \to \mathbb{R}$ be arbitrary. Then define
$$\tilde{X}_t = \int_0^t \theta(u)X(u)\mathrm{d}u.$$
for $t \geq 0$. Show that $\{\tilde{X}_t\}_{t\geq 0}$ is a Gaussian process, and find its mean function and covariance function.

2. For an any $k \geq 1$ and any function $\phi : [0,\infty) \to \mathbb{R}^k$, define the function $\Sigma(s,t) = \langle \phi(s), \phi(t)\rangle$ for all $s,t \geq 0$. Construct a centered Gaussian process $\{X_t\}_{t\geq 0}$ with Σ as its covariance function.

3. Let $\{X_t\}_{t\geq 0}$ be a cosine process with parameters $\sigma^2 > 0$ and $\lambda > 0$, meaning
$$X_t = \varepsilon_1 \cos(\lambda t) + \varepsilon_2 \sin(\lambda t)$$
for all $t \geq 0$, where $\varepsilon_1, \varepsilon_2$ are independent identically-distributed samples from $\mathcal{N}(0,\sigma^2)$. Show for all $k \geq 0$ that the kth derivative process $\{X_t^{(k)}\}_{t\geq 0}$ is a cosine process, and find its parameters.

4. For $v \in \mathbb{R}^2$ and $\theta > 0$, consider the following model for a particle in viscous fluid: We have $X_0 = 0 \in \mathbb{R}^2$, and
$$X_n - X_{n-1} = v - \theta(X_n - X_{n-1}) + \varepsilon_n$$
for $n \geq 1$, where $\varepsilon_1, \varepsilon_2, \ldots$ are independent, identically-distributed samples from $\mathcal{N}(0,\sigma^2)$ for $\sigma^2 > 0$. Find the conditional distribution of X_1 given $\{X_n = x\}$ for $n \geq 0$ and $x \in \mathbb{R}^2$.

5. Let $\{X_n\}_{n\geq 0}$ be a Gaussian random walk, meaning $X_0 = 0$ and $X_n = X_{n-1} + \varepsilon_n$ for $n \geq 1$, where $\varepsilon_1, \varepsilon_2, \ldots$ are independent, identically-distributed samples from $\mathcal{N}(0,\sigma^2)$. Also fix an integer $m \geq 1$.
   (a) Find the conditional distribution of $\{X_n\}_{0\leq n\leq m}$ given $\{X_m = 0\}$.
   (b) Show that $\{X_n - \frac{n}{m}X_m\}_{0\leq n\leq m}$ has the distribution of (a).