STAT GU4207 / GR5207 HOMEWORK #12

SPRING 2025

Assigned: Wed, Apr 16
Due: Fri, Apr 25

1. Show that, if a continuous-time Gaussian process $\{X_t\}_{t\geq 0}$ is mean-square continuous, then the conditional distribution of $\{X_t\}_{t\geq 0}$ given $\{X_r = x\}$ is mean-square continuous for any $r \geq 0$ and $x \in \mathbb{R}$.

2. Let $\{X_t\}_{t\geq 0}$ be a centered Ornstein-Uhlenbeck process, meaning its covariance function is $\Sigma(s,t) = \exp(-\gamma|t-s|)$ for some $\gamma > 0$. Find the conditional distribution of $\{X_t\}_{t\geq 0}$ given $\{X_0 = 0\}$.

3. Let $\{X_t\}_{t\geq 0}$ be a centered Gaussian process with square-exponential covariance function $\Sigma(s,t) = \exp(-\gamma|t-s|^2)$, and take $\beta > 0$. Find a function $v : [0,\infty) \to \mathbb{R}$ such that we have the following convergence of the conditional distribution

$$\left(X_t \,\middle|\, X_0 = 0, X_0' = v(t)\right) \to \mathcal{N}(\beta, 1)$$

as $t \to \infty$. (Recall that, without conditioning, X_t has distribution $\mathcal{N}(0,1)$ for all $t \geq 0$.)

4. Let $\Sigma(s,t) = \exp(-\gamma|t-s|)$ be the Ornstein-Uhlenbeck covariance function, for some $\gamma > 0$. Verify that we have

$$\int_0^\infty \Sigma(s,t)\phi(t)\,\mathrm{d}t = \lambda\phi(s)$$

whenever λ and ϕ are defined as

$$\lambda = \frac{2\gamma}{w^2+\gamma^2}$$

$$\phi(t) = \sqrt{\frac{2w^2}{2\gamma + w^2 + \gamma^2}}\cos(wt) + \sqrt{\frac{2\gamma^2}{2\gamma + w^2 + \gamma^2}}\sin(wt).$$

for some $w \geq 0$.

5. Let $\{X_t\}_{t\geq 1}$ be a centered Gaussian process with covariance function $\Sigma(s,t) = a^2 s^{-2} t^{-2}$ for some $a^2 > 0$. Find the Karhunen-Loève expansion of $\{X_t\}_{t\geq 1}$.