# STAT GU4207 / GR5207 HOMEWORK #13

SPRING 2025

**Assigned: Wed, Apr 23**
**Due: Fri, May 02**

1. Suppose $\{B_t\}_{t\geq 0}$ is a Brownian motion, and define the process $\{\tilde{B}_t\}_{t\geq 0}$ via
$$\tilde{B}_0 = 0 \quad \text{and} \quad \tilde{B}_t = tB_{1/t} \text{ for } t > 0.$$
Show that $\{\tilde{B}_t\}_{t\geq 0}$ is a Brownian motion.

2. Suppose $\{B_t\}_{t\geq 0}$ is a Brownian motion, fix $\gamma > 0$, and define the process $\{X_t\}_{t\geq 0}$ via
$$X_t = e^{-\gamma t} B_{e^{2\gamma t}}.$$
Show $\{X_t\}_{t\geq 0}$ is an Ornstein-Uhlenbeck process.

3. Suppose $\{B_t\}_{t\geq 0}$ is a Brownian motion, fix $x, y > 0$, and write $T := \min\{t \geq 0 : B_t \in \{-x, y\}\}$ for the time of the first exit from the region $(-x, y)$. Use Donsker's theorem to show that
$$\mathbb{P}(B_T = -x) = \frac{y}{x+y} \quad \text{and} \quad \mathbb{P}(B_T = y) = \frac{x}{x+y}.$$

4. Suppose $\{B_t\}_{t\geq 0}$ is a Brownian motion and that $\{N_t\}_{t\geq 0}$ is a Poisson process such that $\{B_t\}_{t\geq 0}$ and $\{N_t\}_{t\geq 0}$ are independent. Now define the process $\{X_t\}_{t\geq 0}$ via
$$X_t = B_{N_t}$$
Show that $\{X_t\}_{t\geq 0}$ has stationary independent increments, but that it is not a Brownian motion.

5. Suppose $\{B_t\}_{t\geq 0}$ is a Brownian motion, and write $T_1 := \min\{t \geq 0 : B_t = 1\}$ for the first passage time to 1. Use the reflection principle to show that $\mathbb{E}[T_1] = \infty$.