STAT GU4207 / GR5207 HOMEWORK #14

SPRING 2025

Assigned: Wed, Apr 30
Due: never

1. Let $\{B_t\}_{t\geq 0}$ be a Brownian motion, and set $T_0 := 0$, and $T_n := \min\{t \geq T_{n-1} : B_t \in \{B_{T_{n-1}} - 1, B_{T_{n-1}} + 1\}\}$ for $n \geq 1$. Then define the process $\{S_n\}_{n\geq 0}$ via $S_n := B_{T_n}$ for $n \geq 0$. Show that $\{S_n\}_{n\geq 0}$ is a simple symmetric random walk.

2. Say that a probability density function $g : \mathbb{R} \to [0, \infty)$ is a *stationary distribution for Brownian motion* if we have
$$\int_{\mathbb{R}} \mathbb{P}(a \leq B_t \leq b \,|\, B_0 = x) g(x)\mathrm{d}x = \int_a^b g(x)\mathrm{d}x$$
for all $t > 0$. Show that there does not exist any stationary distribution for Brownian motion.

3. Suppose that $\{B_t\}_{t\geq 0}$ is a Brownian motion, and that $T > 0$ is fixed. Show that
$$\int_0^T B_t \mathrm{d}t$$
is a Gaussian random variable, and compute its mean and variance.

4. Let $\{B_t\}_{t\geq 0}$ be a Brownian motion, and for $\sigma^2 > 0$, let $\{X_t\}_{t\geq 0}$ be a geometric Brownian motion defined via
$$X_t = \exp\left(-\frac{\sigma^2 t}{2} + \sigma B_t\right).$$
Show that $\mathbb{E}[X_t \,|\, X_s = x] = x$ for all $0 \leq s \leq t$.

5. Let $D \subseteq \mathbb{R}^2$ be the unit disk and let $\partial D \subseteq \mathbb{R}^2$ be its boundary, which is the unit circle. Let $\{B_t\}_{t\geq 0}$ be a 2-dimensional Brownian motion, and define
$$T := \min\{t \geq 0 : B_t \in \partial D\}$$
as the first passage time to ∂D. For an arbitrary function $g : \partial D \to \mathbb{R}$, show the function $h : D \to \mathbb{R}$ defined via
$$h(x) := \mathbb{E}[f(B_T) \,|\, B_0 = x]$$
is a solution to the following partial differential equation, called the *Laplace equation*:
$$\begin{cases} \frac{\mathrm{d}^2 h}{\mathrm{d}x^2}(x, y) + \frac{\mathrm{d}^2 h}{\mathrm{d}y^2}(x, y) = 0 & \text{for } (x, y) \in D \\ h(x, y) = f(x, y) & \text{for } (x, y) \in \partial D \end{cases}.$$