STAT GU4207 / GR5207 HOMEWORK #2

SPRING 2025

Assigned: Wed, Jan 29
Due: Fri, Feb 07

1. Suppose that $\{X_n\}_{n\geq 0}$ is a Markov chain on the state space $\{0,1\}$ with transition matrix given by
$$P := \begin{pmatrix} \alpha & 1-\alpha \\ 1-\beta & \beta \end{pmatrix}.$$
Let $\{Y_n\}_{n\geq 1}$ denote the Markov chain on state space $\{(0,0),(0,1),(1,0),(1,1)\}$ defined as $Y_n = (X_{n-1}, X_n)$. Find the transition matrix of $\{Y_n\}_{n\geq 0}$.

2. Suppose that P is a transition matrix on a finite state space, and that $0 < q < 1$. Define a Markov chain $\{\tilde{X}_n\}_{n\geq 0}$ as follows:
   - First, $\{\tilde{X}_n\}_{n\geq 0}$ holds in its current state for a random amount of time, which follows a geometric distribution on $\{0,1,2,\ldots\}$ with success parameter q.
   - Second, $\{\tilde{X}_n\}_{n\geq 0}$ makes a transition according to P.

   Find the transition matrix $\tilde{P}$ of $\{\tilde{X}_n\}_{n\geq 0}$.

3. Recall that the Ehrenfest gas model on M particles is a Markov chain $\{X_n\}_{n\geq 0}$ on the state space $\{0,1,\ldots,M\}$ with transition matrix
$$P_{i,j} = \begin{cases} \frac{i}{M} \text{ if } & j = i-1 \\ 1-\frac{i}{M} \text{ if } & j = i+1 \\ 0 \text{ otherwise.} & \end{cases}$$
If X_n represents the number of particles in the left of two chambers at time n, then this is the Markov chain corresponding to moving one particle, selected uniformly at random, to the opposite chamber at each time step. Find the transition matrix if, instead, particles in the left chamber are twice as likely to be selected as those in the right chamber.

4. Consider a Markov chain $\{X_n\}_{n\geq 0}$ on the state space $\{0,1,2,3,4\}$ with $X_0 = 0$ and with transition matrix P given by
$$P = \begin{pmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 1/3 & 0 & 2/3 & 0 \\ 0 & 0 & 1/4 & 0 & 3/4 \\ 3/4 & 0 & 0 & 1/4 & 0 \\ 2/3 & 0 & 0 & 0 & 1/3 \end{pmatrix}.$$
Compute $\mathbb{E}[T]$, where $T := \min\{n \geq 1 : X_n = 0\}$ is the time of the first return to 0.

5. Find Markov chains $\{X_n\}_{n\geq 0}$ and $\{Y_n\}_{n\geq 0}$ such that $\{(X_n, Y_n)\}_{n\geq 0}$ is not a Markov chain.