STAT GU4207 / GR5207 HOMEWORK #3

SPRING 2025

Assigned: Wed, Feb 05
Due: Fri, Feb 14

1. Suppose that $\{X_n\}_{n\geq 0}$ is a Markov chain with transition matrix P and limiting distribution π. Compute the value
$$\lim_{n\to\infty} \mathbb{P}(X_n = j \mid X_{n+1} = i).$$

2. Suppose that P is a transition matrix on the state space $\{1, \ldots, M\}$ with the additional property that its columns are probability vectors, that is $\sum_i P_{ij} = 1$ for all j. Show that the uniform distribution $\pi = (\frac{1}{M}, \ldots, \frac{1}{M})$ is a stationary distribution for P.

3. Find all stationary distributions for the following transition matrix:
$$P = \begin{pmatrix} 1/2 & 0 & 0 & 1/2 & 0 \\ 0 & 1/3 & 0 & 0 & 2/3 \\ 0 & 0 & 2/3 & 0 & 1/3 \\ 1/2 & 0 & 0 & 1/2 & 0 \\ 0 & 2/3 & 1/3 & 0 & 0 \end{pmatrix}.$$

4. Fix $\alpha \geq 0$ and consider the following transition matrix on the state space $\{0, 1, 2, \ldots\}$:
$$P_{i,j} = \begin{cases} \frac{1}{1+\alpha} & \text{if } j = i-1 \\ \frac{\alpha}{1+\alpha} & \text{if } j = i+1 \\ 0 & \text{otherwise.} \end{cases} \quad \text{for } i \geq 1, \quad \text{and} \quad P_{0,j} = \begin{cases} \frac{1}{1+\alpha} & \text{if } j = 0 \\ \frac{\alpha}{1+\alpha} & \text{if } j = i \\ 0 & \text{otherwise.} \end{cases}$$
Determine which $\alpha \geq 0$ are such that P admits a stationary distribution; for $\alpha \geq 0$ such that the stationary distribution exists, find the stationary distribution.

5. For positive integers M, N, consider two state spaces $\mathcal{A} := \{a_1, \ldots, a_M\}$ and $\mathcal{B} := \{b_1, \ldots b_N\}$, and let P be the transition matrix on $\mathcal{A} \cup \mathcal{B}$ given by
$$P_{i,j} = \begin{cases} 1/2 & \text{if } j = i \\ 1/2N & \text{if } j \in \mathcal{B} \\ 0 & \text{otherwise} \end{cases} \qquad \text{if } i \in \mathcal{A}$$
and
$$P_{i,j} = \begin{cases} 1/2 & \text{if } j = i \\ 1/2M & \text{if } j \in \mathcal{A} \\ 0 & \text{otherwise} \end{cases} \qquad \text{if } i \in \mathcal{B}.$$
Show that P has a limiting distribution, and find the limiting distribution.