STAT GU4207 / GR5207 HOMEWORK #4

SPRING 2025

Assigned: Wed, Feb 12
Due: Fri, Feb 21

1. Recall that the Wright-Fisher model with M individuals is a Markov chain $\{X_n\}_{n\geq 0}$ on the state space $\{0, 1, \ldots, M\}$ such that the condtional distribution of X_{n+1} given $\{X_n = i\}$ is Binomial$(M, \frac{i}{M})$. Determine the communicating classes of this Markov chain.

2. Show that an irreducible, transient Markov chain cannot have a stationary distribution.

3. A *birth-death process* is a Markov chain on the state space $\{0, 1, 2, \ldots\}$ whose transition matrix has the following form:
$$P = \begin{pmatrix} p_0 & 1-p_0 & 0 & 0 & 0 & \cdots \\ q_1 & 1-q_1-p_1 & p_1 & 0 & 0 & \cdots \\ 0 & q_2 & 1-q_2-p_2 & p_2 & 0 & \cdots \\ 0 & 0 & q_3 & 1-q_3-p_3 & p_3 & \cdots \\ 0 & 0 & 0 & q_4 & 1-q_4-p_4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$
Here, $p_0, p_1, \ldots, q_1, q_2, \ldots$ are non-negative parameters satisfying $p_i + q_i \leq 1$ for all $i \geq 1$ and $p_0 < 1$. Show that P is irreducible and aperiodic if $p_0, p_1, \ldots, q_1, q_2, \ldots$ are all strictly positive.

4. Suppose that $\alpha = (\alpha_0, \alpha_1, \alpha_2, \ldots)$ is a vector with $0 \leq \alpha_i \leq 1$ for all $i \geq 0$ and $0 < \alpha_0 < 1$, and consider a Markov chain on the state space $\{0, 1, 2, \ldots\}$ with transition matrix given by
$$P = \begin{pmatrix} 1-\alpha_0 & \alpha_0 & 0 & 0 & 0 & \cdots \\ 1-\alpha_1 & 0 & \alpha_1 & 0 & 0 & \cdots \\ 1-\alpha_2 & 0 & 0 & \alpha_2 & 0 & \cdots \\ 1-\alpha_3 & 0 & 0 & 0 & \alpha_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$
Determine the α for which this chain is ergodic.

5. Suppose $\{X_n\}_{n\geq 0}$ is a Markov chain where the conditional distribution of X_{n+1} given $\{X_n = i\}$ is Unif$\{0, 1, \ldots, i-1\}$ for $i \geq 1$. Compute $\mathbb{E}[T_0 \,|\, X_0 = i]$ for $i \geq 1$ where $T_0 := \min\{n \geq 1 : X_n = 0\}$.