STAT GU4207 / GR5207 HOMEWORK #6

SPRING 2025

Assigned: Wed, Feb 26
Due: Fri, Mar 07

1. (a) Suppose that N is a non-negative integer-valued random variable satisfying
$$\mathbb{P}(T > m \,|\, T > n) = \mathbb{P}(T > m - n)$$
for all $0 < n < m$. Show that there exists some $0 \leq p \leq 1$ such that T has a $\text{Geo}(p)$ distribution on $\{0, 1, 2, \ldots\}$.
   (b) Suppose that T is a non-negative continuous random variable satisfying
$$\mathbb{P}(T > t \,|\, T > s) = \mathbb{P}(T > t - s)$$
for all $0 < s < t$. Show that there exists some $0 \leq \lambda \leq \infty$ such that T has an $\text{Exp}(\lambda)$ distribution. (Here, we take the convention that a random variable T has distribution $\text{Exp}(0)$ if $\mathbb{P}(T = \infty) = 1$ and has distribution $\text{Exp}(\infty)$ if $\mathbb{P}(T = 0) = 1$.)

2. Fix $\lambda > 0$ and suppose for each $n \geq 1$ that N_n has a $\text{Geo}(\lambda/n)$ distribution on $\{0, 1, 2, \ldots\}$, and that T is a random variable with an $\text{Exp}(\lambda)$ distribution. Show for all $t > 0$ that
$$\mathbb{P}(n^{-1} N_n > t) \to \mathbb{P}(T > t)$$
as $n \to \infty$. In other words, show that $n^{-1}\text{Geo}(\lambda/n)$ converges in distribution to $\text{Exp}(\lambda)$.

3. Let $\{N_t\}_{t \geq 0}$ be a Poisson process of rate λ, fix $\alpha > 0$, and let T be a non-negative continuous random variable whose conditional distribution with respect to the process $\{N_t\}_{t \geq 0}$ is defined via
$$\mathbb{P}(T > t \,|\, N_t = k) = \alpha^k$$
for all $t > 0$ and $k \geq 0$. Find the marginal distribution of T.

4. Let $\{N_t\}_{t \geq 0}$ be a Poisson process of rate λ, and let T be a random variable with distribution $\text{Exp}(\mu)$ which is independent of $\{N_t\}_{t \geq 0}$. Find the distribution of N_T.

5. Let $\{N_t\}_{t \geq 0}$ be a Poisson process of rate λ, and suppose that all arrived units incur a cost $c > 0$ per unit per time until they are removed. Suppose that, in periods of non-random length $L > 0$, all arrived units are removed and a total cost of r is incurred, regardless of the number of arrived units. What period length $L > 0$ minimizes the expected total cost per time, as a function of λ, c, and r?