STAT GU4207 / GR5207 HOMEWORK #7

SPRING 2025

Assigned: Wed, Mar 05
Due: Fri, Mar 12

1. Let $\{N_t\}_{t\geq 0}$ be a Poisson process of rate $\lambda > 0$ and let $S_1, S_2, \dots$ denote its arrival times. Show, for each $t > 0$, that N_t and S_{N_t+1} are independent.

2. Suppose $\{N_t\}_{t\geq 0}$ is a Poisson process of rate $\lambda > 0$ with arrival times $S_1, S_2, \dots$, and define the random varible
$$R_t = \begin{cases} t - S_{N_t} & \text{if } N_t \geq 1 \\ t & \text{if } N_t = 0 \end{cases}$$
for all $t \geq 0$. Compute $\lim_{t\to\infty} \mathbb{E}[R_t]$.

3. Suppose that $\{N_t\}_{t\geq 0}$ is a Poisson process of rate $\lambda > 0$, let $W_1, W_2, \dots$ be IID continuous random variables with density f, and define
$$U_t := \sum_{k=1}^{N_t} \mathbf{1}\{S_k + W_k > t\}.$$
Show that U_t converges to a Poisson distribution as $t \to \infty$, and find its parameter. (In other words, compute the limiting distribution of the $M/G/\infty$ queue.)

4. Let $\{N_t\}_{t\geq 0}$ be a Poisson process of rate $\lambda > 0$, let $Z_1, Z_2 \dots$ be IID positive random variables with $\mathbb{E}[Z_k] = 1$ and $\mathrm{Var}(Z_k) = \sigma^2$, and define
$$V_t := \prod_{k=1}^{N_t} Z_k$$
Compute $\mathrm{Var}(V_t)$.

5. Let $S_1, S_2, \dots \in \mathbb{R}^2$ denote the points of a spatial Poisson point process in $\mathbb{R}^2$ with rate $\lambda > 0$, and let $R_1, R_2, \dots$ be IID positive random variables with a density f. Now let $B_1, B_2, \dots$ be a collection of random circles, where B_k has center S_k and radius R_k for all $k \geq 0$. Show that the total number of circles containing the origin has a Poisson distribution with parameter
$$2\pi\lambda \int_0^\infty r^2 f(r)\,\mathrm{d}r$$
(Hint: If S is a point selected uniformly at random in a circle of center 0 and radius 1, then the norm $\|S\|$ is a continuous random variable with density $2\pi r$.)