STAT GU4207 / GR5207 HOMEWORK #8

SPRING 2025

Assigned: Wed, Mar 12
Due: Fri, Mar 28

1. For $\{N_t\}_{t\geq 0}$ a Poisson process with rate $\lambda > 0$, show $\mathbb{E}\left[S_{N_t+1} - S_{N_t}\right] = \frac{2-e^{-\lambda t}}{\lambda}$.

2. Let $\{N_t\}_{t\geq 0}$ denote a Hawkes process with no base rate and with excitation function $\mu(t) = \alpha e^{-\beta t}$ for $\alpha, \beta > 0$. Show that
$$\mathbb{E}\left[\lim_{t\to\infty} N_t\right] = \begin{cases} \frac{\beta}{\beta-\alpha} & \text{if } \alpha < \beta \\ \infty & \text{if } \alpha \geq \beta. \end{cases}$$

3. Let $\{N_t\}_{t\geq 0}$ denote a renewal process whose interarrival times come from a distribution with density f and cumulative distribution function F. Show that the renewal function $m(t) := \mathbb{E}[N_t]$ satisfies the renewal equation $m(t) = F(t) + \int_0^t m(t-x) f(x)\,\mathrm{d}x$ for $t \geq 0$.

4. A *Yule process of rate* $\beta > 0$ is a continuous-time Markov chain $\{X_t\}_{t\geq 0}$ on the state space $\{1, 2, 3, \ldots\}$ with infinitesimal generator given by
$$Q = \begin{pmatrix} -\beta & \beta & 0 & 0 & \cdots \\ 0 & -2\beta & 2\beta & 0 & \cdots \\ 0 & 0 & -3\beta & 3\beta & \cdots \\ 0 & 0 & 0 & -4\beta & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$
Verify that the transition function $P_{ij}(t) = \binom{j-1}{i-1} e^{-\beta i t}(1-e^{-\beta t})^{j-i}$ has infinitesimal generator Q, and use this to compute $\mathbb{E}[X_t]$.

5. For $M > 0$ and $\gamma > 0$, consider a continuous-time Markov chain $\{X_t\}_{t\geq 0}$ on the state space $\{0, 1, \ldots, M\}$ with infinitesimal generator given by
$$Q = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \gamma & -\gamma & 0 & \cdots & 0 & 0 & 0 \\ 0 & \gamma & -\gamma & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -\gamma & 0 & 0 \\ 0 & 0 & 0 & \cdots & \gamma & -\gamma & 0 \\ 0 & 0 & 0 & \cdots & 0 & \gamma & -\gamma \end{pmatrix}$$
Find the transition function $\{P(t)\}_{t\geq 0}$ for this Markov chain.