STAT GU4207 / GR5207 HOMEWORK #9

SPRING 2025

Assigned: Wed, Mar 26
Due: never

1. Suppose that Q is a generator matrix and that a probability vector π is *reversible for Q* meaning $\pi_i Q_{ij} = \pi_j Q_{ji}$ for all states $i \neq j$. Show that π is stationary for Q.

2. Give an example of a continuous-time Markov chain with a unique stationary distribution that is not a stationary distribution for its embedded discrete-time Markov chain.

3. For $\kappa > 0$, the *Kimura 2-parameter model* is a continuous-time Markov chain on the state space $\{A, G, C, T\}$ with generator matrix given by
$$Q = \begin{pmatrix} -(\kappa+2) & \kappa & 1 & 1 \\ \kappa & -(\kappa+2) & 1 & 1 \\ 1 & 1 & -(\kappa+2) & \kappa \\ 1 & 1 & \kappa & -(\kappa+2) \end{pmatrix},$$
which represents mutation rates for nucleotides appearing in a DNA sequence. Show that this chain has a limiting distribution, and compute it.

4. For $M > 0$, consider the following three-chamber variant of the Ehrenfest gas model:
   - There are M particles distributed into three chambers called left, middle, and right.
   - The left chamber is adjacent to the middle chamber and the middle chamber is adjacent to the right chamber.
   - All particles move chambers according to independent processes of common rate $\lambda > 0$, and, when they move, they move uniformly at random to an adjacent chamber.

   The number of particles in the left chamber is a continuous-time Markov chain denoted $\{X_t\}_{t\geq 0}$. Show that $\{X_t\}_{t\geq 0}$ has a limiting distribution, and compute it.

5. Consider an $M/M/k$ queue with arrival rate $\lambda > 0$ and processing rate $\mu > 0$, for positive integer k. Show that this chain is ergodic if and only if $k\mu < \lambda$.