STAT GU4207 / GR5207 MIDTERM #1

SPRING 2025

1. (10 points) Recall that the Ehrenfest gas model on M particles is a Markov chain $\{X_n\}_{n\geq 0}$ on the state space $\{0, 1, \ldots, M\}$ with transition matrix
$$P_{i,j} = \begin{cases} \frac{i}{M} & \text{if } j = i-1 \\ 1 - \frac{i}{M} & \text{if } j = i+1 \\ 0 & \text{otherwise.} \end{cases}$$
If X_n represents the number of particles in the left of two chambers at time n, then this is the Markov chain corresponding to moving one particle, (selected uniformly at random) to the opposite chamber at each time step. Find the transition matrix if, instead, we move two particles (selected uniformly at random, without replacement) at each time step.

2. (10 points) Suppose that $\{X_n\}_{n\geq 0}$ is a Markov chain on a finite state space and with transition matrix P and limiting distribution π. Compute the value
$$\lim_{n\to\infty} \mathbb{P}(X_{n+1} = X_{n-1}).$$

3. (10 points) Give an example of a Markov chain which is not regular but which has a unique stationary distribution.

4. (10 points) Let $\{X_n\}_{n\geq 0}$ be a Markov chain on the state space $\{0, 1, 2, 3\}$ with transition matrix
$$P = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 1/2 & 0 & 0 & 1/2 \end{pmatrix}$$
Compute $\mathbb{E}[T \mid X_0 = 0]$ where T is the second time at which $\{X_n\}_{n\geq 0}$ is in state 3

5. (10 points) Suppose that $\alpha = (\alpha_0, \alpha_1, \alpha_2, \ldots)$ is a probability vector with $\alpha_0 > 0$ and consider a Markov chain on the state space $\{0, 1, 2, \ldots\}$ with transition matrix given by
$$P = \begin{pmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \cdots \\ 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$
Determine the α for which this chain is ergodic.