STAT GU4207 / GR5207 MIDTERM #2

SPRING 2025

1. (10 points) If $\{N_t\}_{t\geq 0}$ is a Poisson process with rate $\lambda > 0$, compute $\mathrm{Cov}(N_s, N_t)$ for $0 \leq s \leq t$.

2. (10 points) Suppose that $\{N_t\}_{t\geq 0}$ is a Poisson process of rate λ with arrival times $S_1, S_2, \ldots$, and that $H_1, H_2, \ldots$ are random variables that are conditionally independent given $\{N_t\}_{t\geq 0}$ and such that the conditional distribution of H_k given $\{N_t\}_{t\geq 0}$ is $\mathrm{Exp}(S_k)$. Show that
$$\sum_{k=1}^{\infty} \mathbf{1}\{H_k \geq \beta\}$$
is a Poisson random variable with parameter λ/β.

3. (10 points) Suppose that $\{S_1, S_2, \ldots\}$ are the points of a spatial Poisson point process in $\mathbb{R}^2$ with rate $\lambda > 0$, and write $S_k = (x_k, y_k)$ for each $k \geq 1$. Define
$$\tilde{S}_k = (x_k, -y_k)$$
for all $k \geq 1$. Show by definition that $\{\tilde{S}_1, \tilde{S}_2, \ldots\}$ are the points of a spatial Poisson point process in $\mathbb{R}^2$ with rate $\lambda > 0$.

4. (10 points) Let $\{N_t\}_{t\geq 0}$ be a Hawkes process with no base rate and with constant excitation function, that is $\mu(t) = \lambda > 0$ for all $t \geq 0$, and let $S_1, S_2, \ldots$ denote the arrival times of $\{N_t\}_{t\geq 0}$. Compute $\mathbb{E}[S_k]$ and $\mathrm{Var}(S_k)$ for all $k \geq 0$.

5. (10 points) Consider a continuous-time Markov chain $\{X_t\}_{t\geq 0}$ on the state space $\{0, 1, 2, \ldots\}$ with infinitesimal generator given by
$$Q = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & \cdots \\ 1 & -2 & 1 & 0 & 0 & \cdots \\ 1 & 0 & -2 & 1 & 0 & \cdots \\ 1 & 0 & 0 & -2 & 1 & \cdots \\ 1 & 0 & 0 & 0 & -2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$
Compute $\mathbb{E}[T_0 \mid X_0 = 1]$, where $T_0 := \min\{t \geq 0 : X_t = 0\}$ is the time of the first visit to state 0.